Notes on Needham, Chapter 8

G. Palmer, December 21, 2022

P. 98. The derivation at the bottom of the page establishes a point which seems obvious: The angle between ξ and $\in$ shrinks and σ goes to 0 as $\in$ goes to 0. It follows that $\xi \asymp \in$.

P. 99. *Likewise the Taylor expansion of cos x tells us [exercise] that the graph y = cos x has* $k = -1$ *at the point (0,1), and the circle of curvature is therefore the unit circle, centred at the origin.*

The first three derivatives of cos x are

$$f(x) = \cos x = 1$$
$$f'(x) = -\sin x = 0$$
$$f''(x) = -\cos x = -1$$

The Taylor's expansion is

$$f(\epsilon) - f(0) = f'(0)\,\epsilon + \frac{f''(0)\,\epsilon^2}{2!} + \ldots = (0-1)\,\frac{\epsilon^2}{2}$$

Let $\sigma \asymp -1\left(\epsilon^2/2\right)$ (the second term, not the sum). Then by (8.2),

$$\kappa = 2\left(-\epsilon^2/2\right)/\epsilon^2 = -1$$

Then $r = \left|\frac{1}{\kappa}\right|$, so $r = 1$, showing that the circle of curvature is the unit circle.

If we write the curve as $(x,\ \cos x)$, the second derivative at $(x = 0)$ provides a vector $(0,\ -1)$ orthogonal to the curve at (0,1), which has tangent (1, 0). So it is parallel to the radius and pointing from (0, 1) towards the centre of curvature, so the sign of curvature of the cosine is negative at (0, 1). The magnitude of the orthogonal vector is 1, so $k = -1$ and $r = \left|\frac{1}{k}\right| = 1$. One can also graph the cosine to see that it faces downward at (0,1).

P. 100, ¶ 1. *Suppose that the curve C is the graph y = f(x). If the x-axis is drawn parallel to the tangent line* $\mathcal{T}$*, then Taylor's Theorem implies [exercise] that* $\sigma = (1/2)\,f''\left(x_p\right)\epsilon^2$*, where* x_p *denotes the x coordinate of p.*

One way of writing Taylor's theorem is

$$f(\alpha + \delta x) - f(\alpha) = f'(\alpha)\,\delta x + \frac{f''(\alpha)\,\delta x^2}{2!} + \ldots$$

For this example, let $\alpha = x_p$, and let $\delta x = t - x_p = \epsilon$. Then

$$\sigma = f(x_p + \epsilon) - f(x_p) = f'(x_p)\,\epsilon + (1/2)\, f''(x_p)\,\epsilon^2 + \ldots$$

$$= (0 + (1/2)\, f''(x_p)\,\epsilon^2 + \ldots) \quad [f'(x_p) = 0 \text{ because } x \text{ axis parallels } \mathcal{T}.]$$

$$\doteqdot (1/2)\, f''(x_p)\,\epsilon^2$$

P. 100, ¶ 4. ...*We now suppose the x-axis to be drawn horizontally, instead of along the tangent* $\mathcal{T}$. *Taylor's theorem now says that if* $\delta x \equiv x_q - x_p$ then the height of q above $\mathcal{T}$ *is given by* $aq \doteqdot (1/2)\, f''(\delta x)^2$.

Vasco (P.c., VDGF Forum) has provided a derivation which I paraphrase here.

$$f(\alpha + \delta x) - f(\alpha) = f'(\alpha)\,\delta x + \frac{f''(\alpha)\,\delta x^2}{2!} + \ldots \quad \text{(Taylor's)}$$

For this example,

$$qb = f'(p)\,\delta x + (1/2)\, f''(p)\,\delta x^2 + \ldots$$

$ab = f'(p)\,\delta x$ by construction, so

$$aq = qb - ab = (f'(p)\,\delta x + (1/2)\, f''(p)\,\delta x^2 + \ldots) - f'(p)\,\delta x$$

$$= (1/2)\, f''(p)\,(\delta x)^2$$

It is hard, perhaps impossible, to deduce this equation from the geometry alone.

P. 102. Equation. In deriving the equation at the bottom of p. 102, one must sort out the useful equivalents from among other equations. Note that **v** is a vector and v is a real number. All vectors are indicated in boldface.

$$F = m\boldsymbol{A} = m\,\frac{d\boldsymbol{v}}{dt} = \kappa m v^2\,\boldsymbol{N} \qquad (1)$$

Given the third term in (1), it appears that we can further the derivation by finding $\frac{d\boldsymbol{v}}{dt}$ in terms of κ, v, and **N**.

$$\frac{d\boldsymbol{v}}{dt} = \frac{d(v\boldsymbol{T})}{dt} = (0 \cdot \boldsymbol{T} + v\,\dot{\boldsymbol{T}}) = v\,\dot{\boldsymbol{T}} = v\,\frac{d\boldsymbol{T}}{dt} \qquad (2)$$

From (2), it appears that it would be helpful to find $\frac{d\boldsymbol{T}}{\mathrm{dt}}$ in terms of v, κ, and **N**. From (8.5), $\frac{d\boldsymbol{T}}{\mathrm{ds}} = \kappa\boldsymbol{N}$, so we are part way there, and from ¶ -1, p. 102, δs = vδt, which ultimately becomes ds = vdt, so

$$\frac{d\boldsymbol{T}}{\mathrm{ds}} \asymp \frac{d\boldsymbol{T}}{\mathrm{vdt}} = \kappa\boldsymbol{N} \Longrightarrow \frac{d\boldsymbol{T}}{\mathrm{dt}} = \mathrm{v}\kappa\boldsymbol{N}$$

which is just what was needed for (2). Then

$$\boldsymbol{F} = m\,\frac{d\boldsymbol{v}}{\mathrm{dt}} = m\left(v\,\frac{d\boldsymbol{T}}{\mathrm{dt}}\right) = m(v(\mathrm{v}\kappa\boldsymbol{N})) = \kappa \mathrm{mv}^2\,\boldsymbol{N}$$

This is what we wanted. I arrived at it only after a critique of my first attempt, not that it was wrong, but that it could be more direct, because it resorted to new information from [8.6][b].

$$\boldsymbol{F} = m\boldsymbol{A} = m\,\frac{d\boldsymbol{v}}{\mathrm{dt}} = m\left(0\cdot\boldsymbol{T} + v\,\dot{\boldsymbol{T}}\right) = \mathrm{mv}\dot{\psi}\mathbf{N},\text{ where } \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{dt}}$$

From [8.6][b],

$$\delta\varphi = \kappa\mathrm{v}\delta\mathrm{t} \Longrightarrow \frac{\mathrm{d}\varphi}{\mathrm{dt}} = \dot{\varphi} = \kappa\mathrm{v}$$

Then

$$\boldsymbol{F} = \mathrm{mv}\,\dot{\varphi}\,\boldsymbol{N} = \mathrm{mv}(\kappa\mathrm{v})\,\boldsymbol{N} = \kappa\mathrm{mv}^2\,\boldsymbol{N}$$

The alternative derivation supports the construction.

P. 103, ¶ 2. [exercise] By differentiating both sides of... tan $\psi = \left(\frac{\dot{y}}{\dot{x}}\right)$, and using the chain rule, ... obtain (8.6). By "using the chain rule" I am supposing that he means $\frac{d(f(\phi))}{\mathrm{dt}} = \frac{\mathrm{df}}{\mathrm{d}\varphi}\,\frac{\mathrm{d}\varphi}{\mathrm{dt}} = f\,'\,\dot{\varphi}$. I am also using Leibnitz' equation for differentiating quotients. In this note and the following, I assume that $f\,'$ by itself means $\frac{\mathrm{df}}{\mathrm{dx}}$. If f is known to be a function of φ, then $f\,'$ means $\frac{\mathrm{df}}{\mathrm{d}\varphi}$ (Vasco, p.c. VDGAF Forum). Following Newton, derivatives of functions of t are written $\dot{f}$, $\dot{x}$, $\dot{y}$, $\dot{\varphi}$, and so on.

$$\frac{d(\tan\varphi)}{\mathrm{dt}} = \frac{d(\sin\varphi/\cos\varphi)}{\mathrm{dt}} = \frac{\cos\varphi\,\dot{\varphi}\bullet\cos\varphi - \sin\varphi\bullet(-\sin\varphi\,\dot{\varphi})}{\cos^2\varphi} = \frac{\dot{\varphi}}{\cos^2\varphi}$$

From [8.6][b], and letting $\sqrt{\dot{x}^2 + \dot{y}^2} = v$, $\cos\varphi = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\dot{x}}{v}$ and $\delta\varphi \asymp \kappa\mathrm{v}\delta\mathrm{t}$, which implies $\dot{\varphi} \asymp \kappa\mathrm{v}$. Then

$$\frac{d(\tan\varphi)}{\mathrm{dt}} = \frac{\dot{\varphi}}{\cos^2\varphi} = \frac{\kappa\mathrm{v}}{\cos^2\varphi} = \kappa\mathrm{v}\,\frac{v^2}{\dot{x}^2} = \frac{\kappa\mathrm{v}^3}{\dot{x}^2}$$

hence

$$\kappa = \frac{d(\tan\varphi)}{\mathrm{dt}}\ \frac{\dot{x}^2}{v^3} \qquad (1)$$

Now differentiate $\frac{\dot{y}}{\dot{x}}$ to obtain an expression for $\frac{d(\tan\varphi)}{\mathrm{dt}}$ in terms of x and y.

$$\frac{d(\tan\varphi)}{\mathrm{dt}} = d\left(\frac{\dot{y}}{\dot{x}}\right)\Big/\mathrm{dt} = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\dot{x}^2}$$

Then substitute into (1), cancel like terms in top and bottom, and substitute for v.

$$\kappa = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\dot{x}^2}\ \frac{\dot{x}^2}{v^3} = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{v^3} = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}, \text{ which is (8.6)}$$

Pp. 103-104. [exercise] The shaded area of the triangle in [8.6][b] with edge vectors

$$\boldsymbol{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \quad \text{and} \qquad \delta\boldsymbol{v} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}\delta\mathrm{t}$$

has area ultimately equal to half the determinant of the matrix with these two columns:

$$\textit{(area of triangle bounded by velocity vectors)} = (1/2)\left(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}\right)\delta t$$

By p. 28, *VCA*, the cross product of **v** and ***δ*****v** is taken to be the signed area of the parallelogram spanned by vectors **v** and ***δ*****v**.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \times \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} \delta\mathrm{t} = \left(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}\right)\delta\mathrm{t}$$

The two single column vectors of this cross product are multiplied in the same way as the determinant of a single matrix of two columns:

$$\begin{pmatrix} \dot{x} & \delta\mathrm{t}\,\ddot{x} \\ \dot{y} & \delta\mathrm{t}\,\ddot{y} \end{pmatrix} = \left(\dot{x}\,\delta\mathrm{t}\,\ddot{y} - \dot{y}\,\delta\mathrm{t}\,\ddot{x}\right) = \left(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}\right)\delta\mathrm{t}$$

Hence, half that area gives the area of the outside triangle in [8.6][b], which is bounded by **v**, ***δ*****v**, and ***v*** + ***δ*****v**. As δt approaches 0, the area of the bounded outside triangle ultimately becomes the area of the shaded triangle. Thus

$$\frac{1}{2} v^3 \kappa \delta t \simeq \frac{1}{2} \left(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}\right) \delta t$$

Substitute $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ and $\kappa = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}$ on the LHS

$$\frac{1}{2} \sqrt{\dot{x}^2 + \dot{y}^2}^{\,3} \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}} \delta t \simeq \frac{1}{2} \left(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}\right) \delta t$$

Cancel $\sqrt{\dot{x}^2 + \dot{y}^2}^{\,3}$ top and bottom and rearrange first and second derivatives, thereby proving (8.6).

Out[•]=

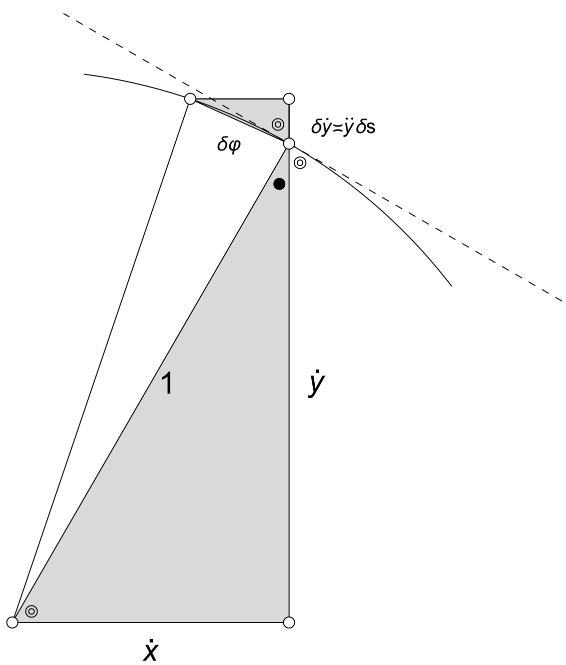

Figure 1. Adapted from [8.7], p. 104

P. 104. [8.7] It was not immediately clear to me why $\delta\dot{y} \simeq \ddot{y}\,\delta s$ or why the two shaded triangles are similar. Vasco (p.c. VDGAF Forum) commented *The reason that* $\delta\dot{y} \simeq \ddot{y}\,\delta s$ *is because by definition* $\delta\dot{y} \simeq \ddot{y}\,\delta t$ *and* $\delta s = \delta t$. The similarity of the shaded triangles is shown in Figure 1. The dotted line is the tangent at the point where the two shaded triangles touch. One can also see that the sides of the small triangle are rotated by $\pi/2$ compared with the corresponding sides of the large triangle.