Needham, VDGAF, Chapter 7, Exercise 9, p. 85.

Palmer, March 23, 2022, 5:35 pm PDT

9. Conformal Curvature Formula. Apply the conformal curvature formula (4.16) to the metric (4.22) of the sphere under stereographic projection, and confirm that $\mathcal{K} = +1\big/R^2$.

$$\mathcal{K} = -\frac{\nabla^2 \ln \Lambda}{\Lambda^2} \qquad (4.16)$$

$$d\,\hat{s} = \frac{2}{1+(r/R)^2}\,\mathrm{ds} \qquad (4.22)$$

From (4.22) let $\Lambda = \frac{2}{1+(r/R)^2}$.

$$\partial_x \Lambda = -\frac{\Lambda^2 x}{R^2}, \qquad \partial_y \Lambda = -\frac{\Lambda^2 y}{R^2}$$

$$\partial_x \ln\Lambda = \frac{1}{\Lambda}\,\partial_x \Lambda = -\frac{\Lambda x}{R^2}, \qquad \partial_y \ln\Lambda = -\frac{\Lambda y}{R^2}$$

$$\partial_x^2 \ln\Lambda = \partial_x\left(-\frac{\Lambda x}{R^2}\right) = -\frac{1}{R^2}\left[(\partial_x \Lambda)\,x + \Lambda\right]$$

$$= -\frac{1}{R^2}\left[\left(-\frac{\Lambda^2 x}{R^2}\right)x + \Lambda\right] = -\frac{1}{R^2}\left[-\frac{\Lambda^2 x^2}{R^2} + \Lambda\right]$$

$$\partial_y^2 \ln\Lambda = -\frac{1}{R^2}\left[-\frac{\Lambda^2 y^2}{R^2} + \Lambda\right]$$

$$\nabla^2 \ln\Lambda = \partial_x^2 \ln\Lambda + \partial_y^2 \ln\Lambda = -\frac{1}{R^2}\left[-\frac{\Lambda^2 x^2}{R^2} + \Lambda - \frac{\Lambda^2 y^2}{R^2} + \Lambda\right]$$

$$= -\frac{1}{R^2}\left[-\frac{\Lambda^2 r^2}{R^2} + 2\Lambda\right]$$

$$\mathcal{K} = -\frac{\nabla^2 \ln\Lambda}{\Lambda^2} = -\left(-\frac{1}{R^2}\left[-\Lambda^2\left(\frac{r}{R}\right)^2 + 2\Lambda\right]\Big/\Lambda^2\right) = \frac{1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + \frac{2}{\Lambda}\right]$$

$$= \frac{1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + 2\times\left(1+\left(\frac{r}{R}\right)^2\right)/2\right] = \frac{1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + \left(1+\left(\frac{r}{R}\right)^2\right)\right]$$

$$= +\frac{1}{R^2} \quad \text{QED}$$

See also, https://gbp.faculty.unlv.edu/mma/VDG-PDF/CHAPTER4/tn.ch4.notes.pdf