Needham, VDGAF, Chapter 7, Exercise 8, p. 84

Palmer, March 22, 2022, 8:45 am PDT

8. Conformal ("Isothermal") Coordinates. Given the metric of a general surface in general (u, v)-coordinates, (4.7),

$$d\hat{s}^2 = A^2\,du^2 + B^2\,dv^2 + 2\,F\,du\,dv,$$

our aim is to find conformal (U, V)-coordinates, such that if $Z = U + iV$, then

$$d\hat{s}^2 = \Lambda^2\left(dU^2 + dV^2\right) = \Lambda^2\,dZ\,d\overline{Z}.$$

Let $M = \sqrt{A^2 B^2 - F^2}$ be the area magnification factor in going from the (u, v)-map to the surface, so that (4.11) becomes $dA = M\,du\,dv$.

(i) Verify that the metric [(4.7)] *has the following complex factorization:*

$$d\hat{s}^2 = \left[A\,du + \tfrac{(F+iM)}{A}\,dv\right]\left[A\,du + \tfrac{(F-iM)}{A}\,dv\right].$$

The factorization of the metric (4.7) can be verified by expanding the complex factorization.

$$d\hat{s}^2 = A^2\,du^2 + (F - iM)\,dudv + (F + iM)\,dudv + \tfrac{F^2+M^2}{A^2}\,dv^2$$

$$= A^2\,du^2 + 2\,F\,dudv + \tfrac{F^2+M^2}{A^2}\,dv^2$$

Since, $M^2 = A^2 B^2 - F^2$,

$$d\hat{s}^2 = A^2\,du^2 + 2\,F\,dudv + \tfrac{F^2+A^2 B^2-F^2}{A^2}\,dv^2$$

$$= A^2\,du^2 + B^2\,dv^2 + 2\,F\,dudv$$

Which is (4.7).

(ii) Suppose that a complex integrating factor Ω can be found such that

$$\Omega\left[A\,du + \tfrac{(F+iM)}{A}\,dv\right] = dU + i\,dV = dZ$$

Deduce that in this case (U, V) are indeed conformal coordinates on the surface

$$d\hat{s}^2 = \Lambda^2\left(dU^2 + dV^2\right) = \Lambda^2\, dZ d\overline{Z}, \text{ where } \Lambda = \frac{1}{|\Omega|}.$$

Since

$$\Omega\left[A\, du + \frac{(F+iM)}{A}\, dv\right] = dU + i\, dV = dZ,$$

$$\left|\Omega\right|^2 \left|A\, du + \frac{(F+iM)}{A}\, dv\right|^2 = \left|dU + i\, dV\right|^2 = dZ d\overline{Z}$$

$$\left|A\, du + \frac{(F+iM)}{A}\, dv\right|^2 = \frac{|dU+i\, dV|^2}{|\Omega|^2} = \Lambda^2\, dZ\, d\overline{Z}$$

Then we can say that for some $ds^2 = dZ\, d\overline{Z}$, there is a $d\hat{s}^2 = \left|A\, du + \frac{(F+iM)}{A}\, dv\right|^2$ such that $d\hat{s}^2 = \Lambda^2\, ds^2$, and this surface is conformal by (4.14) and the first equation in section 4.6, p. 41. The conformal map requires that the expansion factor not depend on the direction γ of dZ emanating from Z (p. 39). That is the case, so (U, V) are conformal coordinates on the surface $d\hat{s}^2 = \Lambda^2\left(dU^2 + dV^2\right) = \Lambda^2\, dZ d\overline{Z}$, where $\Lambda = \frac{1}{|\Omega|}$. That this $d\hat{s}^2$ is in fact the same as that in part (i) can be verified by calculating $\left|A\, du + \frac{(F+iM)}{A}\, dv\right|^2$ or by observing that $\left|A\, du + \frac{(F+iM)}{A}\, dv\right|$ and $\left|A\, du + \frac{(F-iM)}{A}\, dv\right|$ are conjugates, so that

$$\left|A\, du + \frac{(F+iM)}{A}\, dv\right|^2 = \left[A\, du + \frac{(F+iM)}{A}\, dv\right]\left[A\, du + \frac{(F-iM)}{A}\, dv\right].$$

(iii) Use $df = (\partial_u f)\, du + (\partial_v f)\, dv$ to deduce that

$$\partial_u Z = \Omega A \qquad \text{and} \qquad \partial_v Z = \Omega\left[\frac{F+iM}{A}\right].$$

Since $df = (\partial_u f)\, du + (\partial_v f)\, dv$ and $dZ = \Omega\left[A\, du + \frac{(F+iM)}{A}\, dv\right]$ the deduction requires only inspecting the coefficients of du and dv.

(iv) Deduce that

$$[F + iM]\, \partial_u Z = A^2\, \partial_v Z \qquad (1)$$

Substitute for $\partial_u Z$ and $\partial_v Z$.

$$[F + iM]\, \Omega A = A^2\, \Omega\left[\frac{F+iM}{A}\right]$$

Divide through by Ω and simplify RHS:

$$[F + \mathrm{iM}]\, A = [F + \mathrm{iM}]\, A$$

This is what we wanted.

(v) Multiplying both sides of the previous equation by [F-iM], deduce that

$$B^2\, \partial_u Z = [F - iM]\, \partial_v Z \qquad (2)$$

Substitute for $\partial_u Z$ and $\partial_v Z$.

$$B^2\, \Omega\mathrm{A} = [F - \mathrm{iM}]\, \Omega\left[\frac{F+\mathrm{iM}}{A}\right]$$

Divide through by Ω, multiply out RHS and substitute for M^2.

$$B^2\, A = [F - \mathrm{iM}]\left[\frac{F+\mathrm{iM}}{A}\right] = \frac{F^2+A^2\, B^2-F^2}{A} = \frac{A^2\, B^2}{A} = \mathrm{AB}^2$$

This is what we wanted.

(vi) By equating real and imaginary parts, deduce that the rates of change of U can be expressed in terms of rates of change of V (and vice versa) as follows:

$$\partial_u U = \frac{1}{M}\left[A^2\, \partial_u V - F\, \partial_u V\right] \qquad \textit{and} \qquad \partial_v U = \frac{1}{M}\left[F\, \partial_v V - B^2\, \partial_u V\right]$$

$$\partial_u V = \frac{1}{M}\left[F\, \partial_u U - A^2\, \partial_v U\right] \qquad \textit{and} \qquad \partial_v V = \frac{1}{M}\left[B^2\, \partial_u U - F\, \partial_v U\right]$$

Note from *(7) and (8)*, p. 210 of *VCA*, that $\partial_u Z = \partial_u U + i\, \partial_u V$ and $\partial_u Z = \partial_u U + i\, \partial_u V$. Since the partials of U and V are real, each partial of Z has identifiable real and complex components. From part (iv) equation (1),

$$[F + \mathrm{iM}]\, \partial_u Z = A^2\, \partial_v Z$$

This equation can't be used to equate real and imaginary parts, even when expanded, because $\partial_u Z$ and $\partial_v Z$ are both complex, but it can be done if we first substitute the appropriate partials of U and V.

$$F(\partial_u U + i\, \partial_u V) + \mathrm{iM}(\partial_u U + i\, \partial_u V) = A^2(\partial_v U + i\, \partial_v V)$$

$$F\, \partial_u U + \mathrm{iF}\, \partial_u V + i\, M\, \partial_u U - M\, \partial_u V = A^2\, \partial_v U + \mathrm{iA}^2\, \partial_v V \qquad (3)$$

Equate real parts.

$$F\,\partial_u U - M\,\partial_u V = A^2\,\partial_v U$$

Rearrange keeping in mind that we want a V term expressed in terms of two U terms, or vice versa, and divide through by M.

$$\partial_u V = \tfrac{1}{M}\left[F\,\partial_u U - A^2\,\partial_v U\right] \qquad (4)$$

Equate imaginary parts (times i) of (3), rearrange, and divide through by i and M.

$$\mathrm{iF}\,\partial_u V + i\,M\,\partial_u U = \mathrm{iA}^2\,\partial_v V$$

$$\partial_u U = \tfrac{1}{M}\left[A^2\,\partial_v V - F\,\partial_u V\right] \qquad (5)$$

From part (v) equation (2),

$$B^2\,\partial_u Z = [F - \mathrm{iM}]\,\partial_v Z$$

$$B^2(\partial_u U + i\,\partial_u V) = [F - \mathrm{iM}]\,(\partial_v U + i\,\partial_v V)$$

$$B^2\,\partial_u U + \mathrm{iB}^2\,\partial_u V = F\,\partial_v U + \mathrm{iF}\,\partial_v V - \mathrm{iM}\,\partial_v U + M\,\partial_v V$$

$$B^2\,\partial_u U = F\,\partial_v U + M\,\partial_v V \quad \text{(equate real parts)}$$

$$\partial_v V = \tfrac{1}{M}\left[B^2\,\partial_u U - F\,\partial_v U\right] \quad \text{(solve for } \partial_v V\text{)} \qquad (6)$$

$$\mathrm{iB}^2\,\partial_u V = \mathrm{iF}\,\partial_v V - \mathrm{iM}\,\partial_v U \quad \text{(equate imaginary parts times i)}$$

$$\partial_v U = \tfrac{1}{M}\left[-B^2\,\partial_u V\right] = \tfrac{1}{M}\left[-A^2\,\partial_v U\right] \qquad (7)$$

(vii) Granted that $\partial_u \partial_v \Phi - \partial_v \partial_u \Phi = 0$, deduce that $\Phi = U$, and $\Phi = V$ are both solutions of the Beltrami-Laplace Equation:

$$\partial_v\left[\frac{A^2\,\partial_v\Phi - F\,\partial_u\Phi}{M}\right] + \partial_u\left[\frac{B^2\,\partial_u\Phi - F\,\partial_v\Phi}{M}\right] = 0$$

The contents of the brackets correspond in structure to the results (4-7) in part vi. Then mere substitution solves the equation given the condition.

Substitute V for Φ using (4) and (6) to show that U is a solution.

$$\partial_v[\partial_u U] + \partial_u[-\partial_v U] = 0$$

$$\partial_v \partial_u U - \partial_u \partial_v U = 0$$

$$\partial_u \partial_v U - \partial_v \partial_u U = 0 \quad \text{[multiply through by −1]}$$

Substitute U for Φ using (5) and (7) to show that V is a solution.

$$\partial_v[-\partial_u V] + \partial_u[\partial_v V] = 0$$

$$\partial_u \partial_v V - \partial_v \partial_u V = 0 \quad \text{[rearrange]}$$

The condition $\partial_u \partial_v \Phi - \partial_v \partial_u \Phi = 0$ is met for both U and V.

(viii) Let (x, y) be a second pair of conformal coordinates, so that

$$d\hat{s}^2 = \lambda^2 (dx^2 + dy^2).$$

By setting u = x and v = y, deduce that A = B = λ, F = 0, and M = λ². From (vi), deduce that the local mapping between the (x + iy)–plane and the (U + iV)–plane is an amplitwist, characterized by the celebrated Cauchy-Riemann equations:

$$\partial_x U = \partial_y V \qquad \text{and} \qquad \partial_y U = -\partial_x V.$$

(x, y) and (U, V) are two pair of conformal coordinates on a surface $\mathcal{S}$, but in this case the surface is $\mathbb{C}$ or a copy of $\mathbb{C}$. Consider the general equation for the metric on a surface

$$d\hat{s}^2 = A^2\, du^2 + B^2\, dv^2 + 2F\, du dv$$

For this to equal $d\hat{s}^2 = \lambda^2 (du^2 + dv^2)$, F must equal 0. One can write:

$$\lambda^2 (du^2 + dv^2) = A^2\, du^2 + B^2\, dv^2 \Longrightarrow \lambda^2 = \frac{A^2\, du^2 + B^2\, dv^2}{du^2 + dv^2}$$

The derivatives can only cancel out leaving a scalar if the numerator can be factored to extract a scalar component, say Λ. It can only be factored if $A^2 = B^2$, which implies A = ±B.

$$\lambda^2 = \frac{\Lambda^2 (du^2 + dv^2)}{du^2 + dv^2} = \Lambda^2, \text{ where } \Lambda = \pm A \text{ or } \pm B.$$

Since we are only interested in positive scalars, we can write $A = B = \lambda$, and $F = 0$. Since F = 0, one can write $M = \sqrt{A^2 B^2} = \sqrt{\lambda^4} = \lambda^2$, and any term with factor F vanishes. After substituting u and v for x and y, then from (vi) write:

$$\partial_u U = \frac{1}{M}\left[A^2 \partial_v V - F \partial_u V\right] = \frac{1}{\lambda^2}\left[\lambda^2 \partial_v V\right] = \partial_v V$$

and

$$\partial_v U = \frac{1}{M}\left[F \partial_v V - B^2 \partial_u V\right] = \frac{1}{\lambda^2}\left[-\lambda^2 \partial_u V\right] = -\partial_u V$$

This is what we wanted. The satisfaction of the Cauchy–Riemann equations confirms that the derivative of the function from coordinates (u, v) to (U, V) is an amplitwist (VCA, pp. 209-210). The derivative specifies the *local* [infinitesimal] *mapping between the* $(x + iy)$*–plane and the (*$U + iV$*)–plane.*

(ix) Deduce that in this case the Beltrami-Laplace Equation reduces to the Laplace Equation:

$$\partial_x^2 \Phi + \partial_y^2 \Phi = 0.$$

Reduce the Beltrami-Laplace equation by substituting for A, B, and F as in (viii).

$$\partial_v\left[\frac{A^2 \partial_v \Phi - F \partial_u \Phi}{M}\right] + \partial_u\left[\frac{B^2 \partial_u \Phi - F \partial_v \Phi}{M}\right] = \partial_v\left[\frac{\lambda^2 \partial_v \Phi - F \partial_u \Phi}{\lambda^2}\right] + \partial_u\left[\frac{\lambda^2 \partial_u \Phi - F \partial_v \Phi}{\lambda^2}\right]$$

$$= \partial_v\left[\frac{\lambda^2 \partial_v \Phi - 0}{\lambda^2}\right] + \partial_u\left[\frac{\lambda^2 \partial_u \Phi - 0}{\lambda^2}\right] = \partial_v\left[\partial_v \Phi\right] + \partial_u\left[\partial_u \Phi\right]$$

$$= \partial_u^2 \Phi + \partial_v^2 \Phi = 0$$

Then set u = x and v = y as in (viii) to obtain the Laplace Equation.