Needham, VDGAF, Chapter 7, Exercise 7, p. 83
Palmer, February 18, 2022, 4:30 pm PST

Out[2061]=

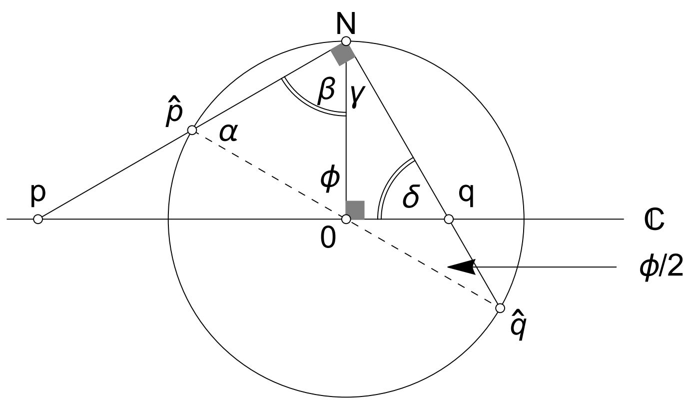

Figure 1. Antipodal points. From p. 83, VDGAF

7. Antipodal Points. The figure below shows a vertical cross section of the Riemann sphere Σ through $\hat{p}$ and the antipodal point $\hat{q}$. Show that the triangles p0N and N0q are similar. Deduce (4.27).

$$\alpha = \pi - (\pi/2 + \phi/2) = \pi/2 - \phi/2$$

$$\beta = \pi - (\alpha + \phi) = \pi - [(\pi/2 - \phi/2) + \phi] = \pi/2 - \phi/2$$

$$\gamma = \pi/2 - \beta = \pi/2 - (\pi/2 - \phi/2) = \phi/2$$

$$\delta = \pi - (\pi/2 + \gamma) = \pi/2 - \phi/2$$

This shows that $\beta = \delta$ in Figure 1. Since triangles p0N and N0q are both right triangles, they are similar by virtue of having two angles in common. Since the Riemann sphere has radius 1, using the similar triangles, one can write

$$\frac{|q|}{1} = \frac{1}{|p|} \quad \Longrightarrow \quad |q| = \frac{1}{|p|}$$

The theta angle is unknown, but since $\hat{p}$ and $\hat{q}$ are antipodal, their images on $\mathbb{C}$ have angles that differ by π. Then if $p = \left| p \right| e^{i\theta}$, q must be $q = \frac{1}{|p|}\, e^{i(\theta+\pi)} = -\frac{1}{|p|}\, e^{i\theta} = -\frac{1}{\left|p\right| e^{-i\theta}} = -\frac{1}{\bar{p}}$, so it follows that $q = -\frac{1}{\bar{p}}$.