Needham, VDGAF, Chapter 7, Exercise 6, p. 83

Palmer, March 1, 2022, 8:55 PST

6. Stereographic metric. Suppose that in [4.10] we choose $\delta\,\hat{s}$ due north (instead of the illustrated choice of due west). Show that leads to the same metric (4.22), and do so in two ways:

Out[∘]=

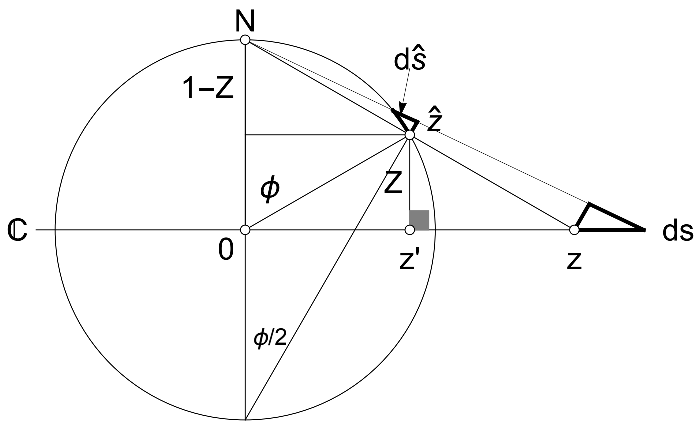

Figure 1. Calculated metric.
dŝ is on sphere.

(4.22) $ds = \frac{2}{1+(r/R)^2}\,ds$,

where R is the radius of the sphere, and $r = |\,z\,| = \sqrt{x^2 + y^2}$.

(i) by calculation

From (4.25),

$$X^2 = \frac{4\,x^2}{\left(1+r^2\right)^2}, \quad Y^2 = \frac{4\,y^2}{\left(1+r^2\right)^2}, \quad Z^2 = \frac{\left(r^2-1\right)^2}{\left(1+r^2\right)}$$

From inspection of Figure 1 we can see that, *ultimately*, the two small triangles at $\hat{z}$ and z have proportional corresponding sides, because the sides of the small triangle at $\hat{z}$ are governed by the ratio of $[N\,\hat{z}]$ to [Nz] and the movement of z. Since both are also right triangles, they are ultimately similar.

$$\frac{d\,\hat{s}}{\text{ds}} \asymp \frac{[N\,\hat{z}]}{[\text{Nz}]}$$

$$d\,\hat{s} \asymp \frac{[N\,\hat{z}]}{[\text{Nz}]}\,\text{ds}$$

Because triangles N0z and $N\hat{z}S$ are similar by virtue of two equal angles,

$$\frac{[N\hat{z}]}{2} = \frac{1}{[\mathrm{Nz}]} \Longrightarrow [\mathrm{Nz}] = \frac{2}{[N\hat{z}]}$$

$$d\,\hat{s} \simeq \frac{[N\hat{z}]}{[\mathrm{Nz}]}\,\mathrm{ds} \Longrightarrow d\,\hat{s} = \frac{[N\hat{z}]}{2/[N\hat{z}]}\,\mathrm{ds} = \frac{[N\hat{z}]^2}{2}\,\mathrm{ds}$$

Note that $|\,z'\,|^2 = X^2 + Y^2$.

$$[N\hat{z}]^2 = (1 - Z)^2 + X^2 + Y^2$$

$$= \left(1 - \frac{r^2-1}{1+r^2}\right)^2 + \left(\frac{2\,x}{1+r^2}\right)^2 + \left(\frac{2\,y}{1+r^2}\right)^2$$

$$= \frac{4}{1+r^2}$$

$$d\,\hat{s} = \frac{[N\hat{z}]^2}{2}\,\mathrm{ds} = \frac{2}{1+r^2}\,\mathrm{ds}$$

Which is equivalent to (4.22) for a sphere of radius 1.

This answer also shows a way to an answer using (4.26).

$$\frac{dz}{d\phi} = \frac{d\cot(\phi/2)\,e^{i\theta}}{d\phi} = \frac{d\left(\cos(\phi/2)/\sin(\phi/2)\,e^{i\theta}\right)}{d\phi}$$

$$\frac{dz}{d\phi} = \frac{-\sin^2(\phi/2)-\cos^2(\phi/2)}{\sin^2(\phi/2)}\,e^{i\theta} = -\frac{e^{i\theta}}{2\sin^2(\phi/2)}$$

Let $ds = |\,dz|$. From the first answer,

$$[N\hat{z}]^2 = \frac{4}{1+r^2}$$

Then

$$sin(\phi\,/2) = \frac{[N\hat{z}]}{2}$$

$$\frac{ds}{d\phi} = \frac{1}{2\sin^2(\phi/2)} = \frac{1}{2\left([N\hat{z}]^2/4\right)} = \frac{2}{\frac{4}{1+r^2}} = \frac{1+r^2}{2}$$

$$d\hat{s} = R \mid d\phi \mid = \mid d\phi \mid$$

$$\frac{ds}{d\hat{s}} = \frac{1+r^2}{2}$$

$$d\hat{s} = \frac{2}{1+r^2}\ \mathrm{ds}$$

(i) by geometry

Refer to [4.10] and [4.11]. Choose the direction of radius $\delta\hat{s}$ emanating from $\hat{z}$ to be vertical along a great circle of longitude. Under stereographic projection, this circle of [longitude] *undergoes a uniform expansion to yield an origin-centred circle through z, with δs pointing along it. As $\hat{z}$ rotates around its circle of* [longitude]*, z rotates with it, and their motions are proportional to their distances from N. Thus,*

$$\frac{\delta\hat{s}}{\delta\mathrm{s}} = \frac{N\hat{z}}{\mathrm{Nz}}.$$

When $\delta\hat{s}$ is due north it points along a meridian of Σ *and δs points at the origin.* Because stereographic projection is conformal and circles project to circles, and because the distances [zN], [$\hat{z}$N], r, and R remain the same, the ratio $\delta\hat{s}/\delta$s on p. 46 still holds locally. Then,

$$[\mathrm{Nz}]^2 \asymp \frac{2R^2\,\delta\mathrm{s}}{\delta\hat{s}}$$

One can still apply Pythagora's theorem to the triangle N0z, so it is still the case that $[\mathrm{Nz}]^2 = R^2 + r^2$ and,

$$\frac{2R^2\,\delta\mathrm{s}}{\delta\hat{s}} \asymp R^2 + r^2$$

$$\delta\hat{s} \asymp \frac{2R^2\,\delta\mathrm{s}}{R^2+r^2} = \frac{2R^2\,\delta\mathrm{s}}{R^2\left(1+(r/R)^2\right)} = \frac{2}{1+(r/R)^2}\ \delta\mathrm{s}$$

So one can write

$$d\hat{s} = \frac{2}{1+(r/R)^2}\ \mathrm{ds} \qquad (2)$$

This shows that solving in terms of geometry leads to the same metric.