Needham, VDGAF, Chapter 7, Exercise 5, p. 83
Palmer, February 5, 2022, 1:40 pm PST

5. Angular Excess of n-gon. Show that the internal angles of a Euclidean n-gon sum to (n-2)π, thereby justifying the definition of angular excess as $\mathcal{E} \equiv$ *(Angle Sum) – (n – 2)π.*

For a regular n-gon, there are n internal angles, each of size $2\pi/n$. Therefore the internal angles at the vertices each have size $(\pi - 2\pi/n)/2$. Since there are 2n of these

$$\Sigma_{\text{internal angles}} = 2\,n\,((\pi - 2\pi/\text{n})/2) = (\text{n} - 2)\,\pi$$

Now take $\triangle_p$ *to be a small n-gon containing a point p on a curved surface, and divide it into triangles by joining p to the vertices. Deduce that (2.1) applies without change.*

$$\mathcal{K} = \lim_{\triangle_p \to p} \frac{\mathcal{E}(\triangle_p)}{\mathcal{A}(\triangle_p)} \qquad (2.1)$$

$\triangle_p$ refers to a "small geodesic triangle ... containing p". Since p lies in the boundary of every triangle in the n-gon, the n triangles must each have the curvature described by (2.1), even though they may be of different sizes and shapes. If $\triangle_p \to p$ for all the triangles simultaneously, they all approximate the same ratio of $\mathcal{E}$ and $\mathcal{A}$, for according to Needham, *this limit exists, independently of the shape of the triangle and the precise manner in which it shrinks*.Then (2.1) applies without change to any of the triangles.