Needham, VDGAF, Chapter 7, Exercise 3, p. 83

Palmer, February 5, 2022, 10:30 am PST

4. Central Projection. Apply the curvature formula (4.10) to the metric (4.2) of the sphere under central projection, and confirm that $\mathcal{K} = +1/R^2$.

Let $\rho = 1 + \left(\frac{r}{R}\right)^2$.

$$\mathcal{K} = -\frac{1}{\mathrm{AB}}\left(\partial_v\left[\frac{\partial_v A}{B}\right] + \partial_u\left[\frac{\partial_u B}{A}\right]\right) \qquad (4.10)$$

$$d\,\hat{s}^2 = \frac{1}{\rho}\left(\frac{\mathrm{d}r^2}{\rho} + r^2\,\mathrm{d}\theta^2\right) \qquad (4.2)$$

Let $d\,\hat{s}_1^2 = \frac{\mathrm{d}r^2}{\rho^2}$ and let $d\,\hat{s}_2^2 = \frac{r^2\,\mathrm{d}\theta^2}{\rho}$.

Then $d\,\hat{s}_1 = \frac{\mathrm{d}r}{\rho}$ and $d\,\hat{s}_2 = \frac{r\,\mathrm{d}\theta}{\sqrt{\rho}}$.

Let $A = \frac{\partial \hat{s}_1}{\partial r} = \frac{1}{\rho}$ and let $B = \frac{\partial \hat{s}_2}{\partial \theta} = \frac{r}{\sqrt{\rho}}$. Apply (4.10) substituting r and θ for u and v respectively.

$$\mathcal{K} = -\frac{1}{\frac{1}{\rho}\cdot\frac{r}{\sqrt{\rho}}}\left(\partial_\theta\left[\frac{\partial_\theta\left(\frac{1}{\rho}\right)}{\frac{r}{\sqrt{\rho}}}\right] + \partial_r\left[\frac{\partial_r\left(\frac{r}{\sqrt{\rho}}\right)}{\frac{1}{\rho}}\right]\right)$$

$$= -\frac{\rho^{3/2}}{r}\left(\partial_\theta\left[\frac{\sqrt{\rho}\;\partial_\theta\left(\frac{1}{\rho}\right)}{r}\right] + \partial_r\left[\rho\,\partial_r\left(\frac{r}{\sqrt{\rho}}\right)\right]\right)$$

$$= -\frac{\rho^{3/2}}{r}\left(\partial_\theta[0] + \partial_r\left[\rho\,\frac{R^4\,\sqrt{\rho}}{R^4\,\rho^2}\right]\right)$$

$$= -\frac{\rho^{3/2}}{r}\,\partial_r\left[\frac{1}{\sqrt{\rho}}\right]$$

$$= -\frac{\rho^{3/2}}{r}\left(-\frac{r}{\rho^{3/2}\,R^2}\right)$$

$$= +\frac{1}{R^2} \qquad \text{QED}$$