Needham, VDGF, Chapter 7, Exercise 34, p. 93.
G. Palmer, December 7, 2022, 10:15 am PT

34. Recall from Special Relativity that a "boost" along the Z-axis yields the Lorentz transformation formulas.

$$\tilde{T} = \frac{T+vZ}{\sqrt{1-v^2}}, \qquad \tilde{X} = X, \qquad \tilde{Y} = Y, \qquad \tilde{Z} = \frac{Z+vT}{\sqrt{1-v^2}}.$$

Comment: I don't find a passage where it was explained how a boost yields the Lorentz transformation formulas, but the "boost" is explained on p. 78.

(i) Show that this transformation can be rewritten as

$$\tilde{T} + \tilde{Z} = \rho(T + Z), \qquad \tilde{X} = X, \qquad \tilde{Y} = Y, \qquad \tilde{T} - \tilde{Z} = (1/\rho)(T - Z),$$

where

$$\rho = \sqrt{\frac{1+v}{1-v}}$$

$$\tilde{T} + \tilde{Z} = \frac{T+vZ}{\sqrt{1-v^2}} + \frac{Z+vT}{\sqrt{1-v^2}} = \frac{(1+v)(T+Z)}{\sqrt{1-v^2}} = \frac{\sqrt{1+v}\,\sqrt{1+v}\,(T+Z)}{\sqrt{1+v}\,\sqrt{1-v}} = \sqrt{\frac{1+v}{1-v}}\,(T+Z) = \rho(T+Z)$$

The second and third equations are simply copied over. The third is calculated like the first, but with $(1/\rho)$ and with negative $\tilde{Z}$ and Z.

(ii) Deduce from (6.19) that this boost can be represented by the spin transformation

$$\begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1/\sqrt{\rho} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

Arrange the boosted transformation in a matrix and invert, meaning express the space time matrix in terms of projective coordinates.

$$\begin{pmatrix} \rho\,(T+Z) & X + \mathrm{i}Y \\ X - \mathrm{i}Y & \left(1/\sqrt{\rho}\right)(T-Z) \end{pmatrix} = 2\begin{pmatrix} \rho\,\zeta_1\,\overline{\zeta_1} & \zeta_1\,\overline{\zeta_2} \\ \zeta_2\,\overline{\zeta_1} & \left(1/\sqrt{\rho}\right)\zeta_2\,\overline{\zeta_2} \end{pmatrix} \qquad \text{(1), from equation above (6.17)}$$

Use the given equation for spin transformation to find $\tilde{\zeta}_1$ in terms of ζ and $\tilde{\zeta}_2$ in terms of ζ_2, as at the top of p. 77. Let

$$[M] = \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1\big/\sqrt{\rho} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1\big/\sqrt{\rho} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\rho}\,\zeta_1 \\ \left(1\big/\sqrt{\rho}\right)\zeta_2 \end{pmatrix}$$

$$\Longrightarrow \zeta_1 = \frac{\tilde{\zeta}_1}{\sqrt{\rho}}, \quad \zeta_2 = \sqrt{\rho}\,\tilde{\zeta}_2, \quad \overline{\zeta_1} = \frac{\overline{\tilde{\zeta}_1}}{\sqrt{\rho}}, \quad \overline{\zeta_2} = \sqrt{\rho}\,\overline{\tilde{\zeta}_2}$$

Then rewrite (1) with appropriate substitutions. It appears that the spin transformation and subsequent substitutions of the boosted values accomplish the $[M]^*$ of $[M][ST][M]^*$, where ST = SPACETIME. In this case $[M]^* = [M]$.

$$\begin{pmatrix} \tilde{T} + \tilde{Z} & \tilde{X} + i\,\tilde{Y} \\ \tilde{X} - i\,\tilde{Y} & \tilde{T} - \tilde{Z} \end{pmatrix} = 2 \begin{pmatrix} \rho\left(\frac{\tilde{\zeta}_1}{\sqrt{\rho}}\right)\left(\frac{\overline{\tilde{\zeta}_1}}{\sqrt{\rho}}\right) & \left(\frac{\tilde{\zeta}_1}{\sqrt{\rho}}\right)\left(\sqrt{\rho}\,\overline{\tilde{\zeta}_2}\right) \\ \left(\sqrt{\rho}\,\tilde{\zeta}_2\right)\left(\frac{\overline{\tilde{\zeta}_1}}{\sqrt{\rho}}\right) & (1/\rho)\left(\sqrt{\rho}\,\tilde{\zeta}_2\right)\left(\sqrt{\rho}\,\overline{\tilde{\zeta}_2}\right) \end{pmatrix}$$

$$= 2 \begin{pmatrix} \tilde{\zeta}_1\,\overline{\tilde{\zeta}_1} & \tilde{\zeta}_1\,\overline{\tilde{\zeta}_2} \\ \tilde{\zeta}_2\,\overline{\tilde{\zeta}_1} & \tilde{\zeta}_2\,\overline{\tilde{\zeta}_2} \end{pmatrix} = 2 \begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix} \begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix}^* \qquad (2)$$

Matrix (2) has the same structure as the unboosted null matrix on the RHS of (6.17), but the projective coordinates have the tilde, showing that the velocity has been boosted (that the matrix has undergone the $[M][ST][M]^*$ transformations). The determinant of (2) is a null spacetime interval and (2) is a null vector on a light ray.

(iii) *Thereby confirm the claim made in [6.5b] and (6.21).*

The fact that ρ does not appear in matrix (2) confirms that the boost on the Z axis conforms to the general null vector. A boost (movement) on the Z axis of a Riemann sphere is a hyperbolic transformation on the complex plane (p. 78), thus confirming *[6.5b]*. By part (i), it also confirms (6.21).

Out[2069]=

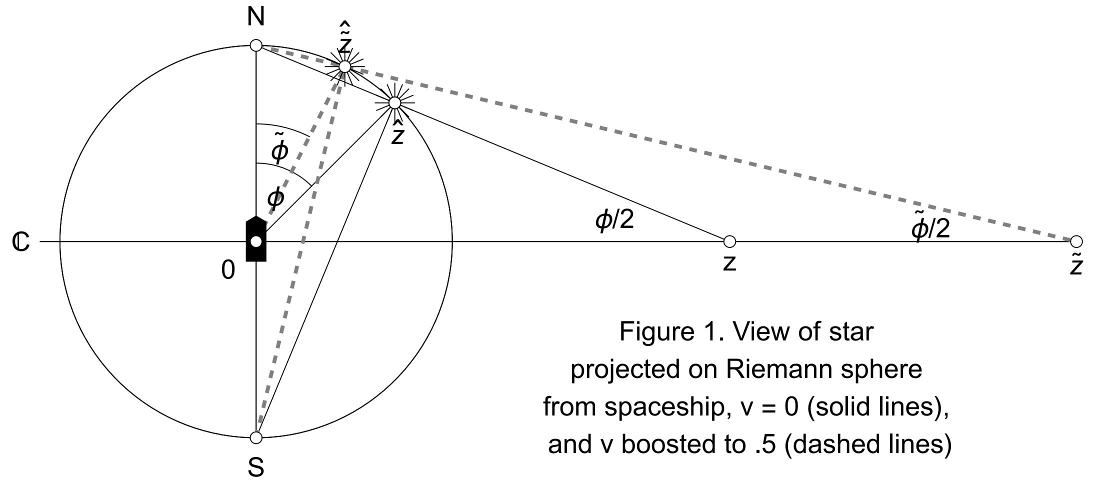

Figure 1. View of star projected on Riemann sphere from spaceship, v = 0 (solid lines), and v boosted to .5 (dashed lines)

(iv) If (θ, ϕ) is the apparent direction of a star before you fire your spaceship engines, deduce that after you have boosted to speed v in the direction $\phi = 0$, the new direction $(\theta, \tilde{\phi})$ of the star is given by

$$cot\,\frac{\tilde{\phi}}{2} = \rho\, cot\,\frac{\phi}{2}$$

The angles in Figure 1 are based on the similarity of the two right triangles having angles of $\phi/2$ and $\tilde{\phi}/2$ at z and $\tilde{z}$ with the two right triangles with angles of $\phi/2$ and $\tilde{\phi}/2$ at S. That in turn follows from the half-angle relationship of right triangles based on a diameter with whole angles at the origin of triangles based on a radius. Let the Riemann sphere stand for the celestial sphere. The perceived angles at the origin $0 \le \phi, \tilde{\phi} \le \pi/2$ of the star from the trajectory of the starship on the Z-axis define points on the Riemann sphere. In figure 1, $\cos(\phi/2)$ represents the projection onto the complex plane of the perceived star in the direction ϕ of $\hat{z}$ (at the angle θ from the X axis) with unfired engines. A projected point on ℂ pertaining to the boosted starship with engines fired can be given as $\cot(\tilde{\phi}/2)$. The boost, ρ, varies non-linearly from 1 to ∞ with increasing v, where $0 \le v \le 1$. Limiting the problem to a star in quadrant I, we can deduce the following:

$1 \le \rho \le \infty$ for all v
$0 \le \phi \le \pi \implies 0 \le \phi/2 \le \pi/2$
$\rho \cot(\phi/2) > \cos(\phi/2)$ and positive for all ϕ

Since $\rho \cot(\phi/2) > \cos(\phi/2)$ and positive for all ϕ, a spaceship at the origin with boosted velocity will have higher values of $\cot(\tilde{\phi}/2)$ than an unboosted ship. Hence, $\cot(\tilde{\phi}/2) = \rho \cot(\phi/2) > \cot(\phi/2)$.

The initial equation can be rewritten as

$$\sqrt{1-v}\,\cot\left(\tilde{\phi}/2\right) = \sqrt{1+v}\,\cot(\phi/2)$$

Hence, one might look for sides $\sqrt{1-v}$ and $\sqrt{1+v}$ on similar or otherwise comparable triangles.

I don't believe I have adequately answered part (iv), but I have no further inspiration for the time being. I will have to return to it later.

(v) Deduce that the star field appears to crowd together towards the north pole, i.e. towards your direction of travel.

By Figure 1, one can see that where $|\tilde{z}| = \cot\left(\tilde{\phi}/2\right) > \left|z\right| = \cot(\phi/2)$, it follows from the construction that $\tilde{\phi} < \phi$. As the ship approaches the north pole, the image of the star climbs the surface of the Riemann sphere along a meridian defined by θ (p. 78, [6.5b]). Thus, the image of the star also approaches the pole, so multiple stars on different meridians would appear to crowd together.

(vi) What happens to the star field as the speed of your spacecraft approaches the speed of light? (!)

$\tilde{\phi}$ shrinks and ultimately vanishes in a singularity if v reaches 1, the speed of light. The star field also vanishes in the singularity.