Needham, VDGF, Chapter 7, Exercise 33, p. 92.
G. Palmer, November 3, 2022, 11:15 am PDT

Out[646]=

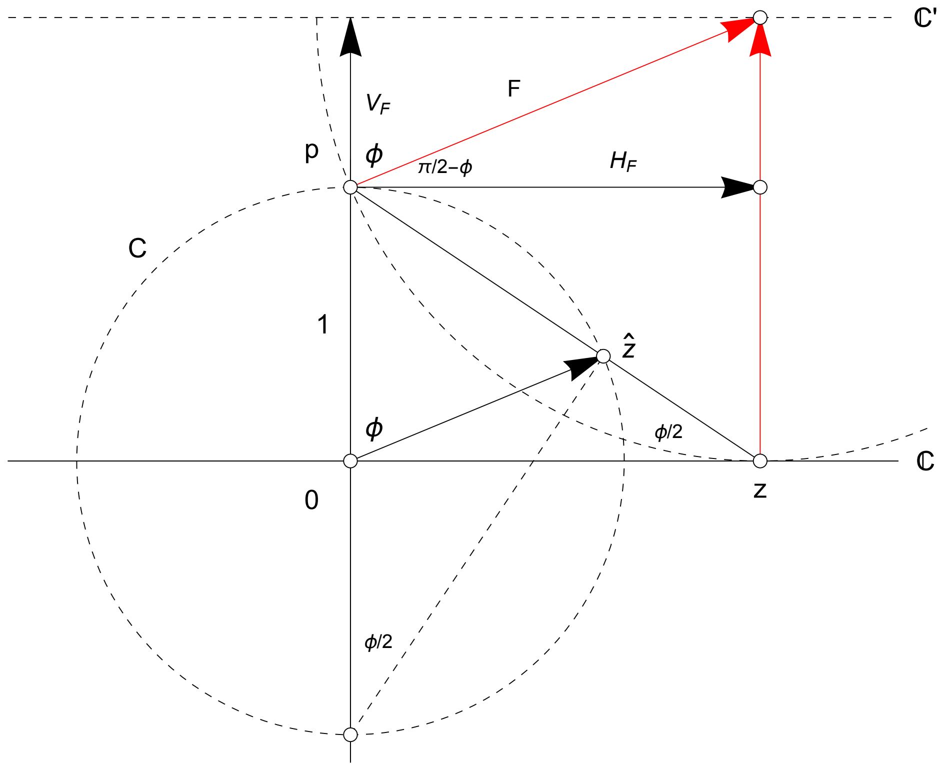

Figure 1. Complex numbers label light rays

33. Penrose's Direct Labeling of Light Rays with Complex Numbers....Let the point p be one unit vertically above the origin of the (horizontal) complex plane. Now imagine that, simultaneously, p emits a flash, and ℂ *begins to travel straight up (in the direction of* ϕ *= 0) at the speed of light (=1) towards p. (You may imagine that the whole of* ℂ flashed, creating a plane wave.) *Decompose the velocity of the photon F emitted by p in the direction* (θ, ϕ) *into components perpendicular and parallel to* ℂ. *Hence, find the time at which F hits* ℂ. *Deduce that F hits* ℂ *at the point* $z = \cot(\phi/2)\, e^{i\theta}$. *Thus, Penrose's construction is equivalent to stereographic projection.*

See [4.12][b], p. 48 and (4.26), p. 49 for background and the template for this exercise. As with the figures [4.12]([a] and [b]), Figure 1 represents a cross section of Σ and ℂ taken through p and $\hat{z}$. Let ℂ' be the moving ℂ-plane, or alternatively, the plane wave. Let F be the vector of the photon, and let V_F and H_F be the vertical and horizontal components of F.

From the exercise, one can deduce the following: Since both the photon and the plane wave move at the same speed, when they meet, their elapsed time is equal and the distance they traversed is equal. Since distance is speed times time, and s = 1, the elapsed time equals the distance. Since the photon must travel in the direction of ϕ, it is plotted emerging from p in a direction parallel to the vector $\hat{z}$. We know from this construction as shown in [4.12][b] and stated in (4.26) (p. 49), that $z = \cot(\phi/2)\, e^{i\theta}$ and $|z| = \cot(\phi/2)$.

(d = s t), where d = distance, s = speed, and t = time

Then the time at which F hits $\mathbb{C}$ is abs(F):

$$t = \text{abs}(F) \;=\; R = 1 + R\sin(\pi/2 - \phi)$$

so

$$R - R\sin(\pi/2 - \phi) \;=\; 1$$

$$R \;=\; \frac{1}{1-\sin(\pi/2-\phi)}$$

$$t \;= R = \;1 + \frac{\sin(\pi/2-\phi)}{1-\sin(\pi/2-\phi)} \;=\; \frac{1-\sin(\pi/2-\phi)+\sin(\pi/2-\phi)}{1-\sin(\pi/2-\phi)} \;=\; \frac{1}{1-\sin(\pi/2-\phi)}$$

One can also write

$$\text{t} = \text{R} = R\,\sqrt{\sin^2(\pi/2 - \phi) + \cos^2(\pi/2 - \phi)} \;=\; \frac{1}{1-\sin(\pi/2-\phi)}$$

One can also write $R\cos(\pi/2 - \phi) = \cot(\phi/2)$. Hence $R \;=\; \cot(\phi/2)/\cos(\pi/2 - \phi) = \cot(\phi/2)/\sin\phi$.

Deduce that F hits $\mathbb{C}$ *at the point* $z = cot(\phi\ /2)\, e^{i\theta}$. *Thus, Penrose's construction is equivalent to stereographic projection.*

This deduction seems harder to make. I think one must assume that the moving $\mathbb{C}$' is still $\mathbb{C}$, so the horizontal component of F on $\mathbb{C}$ equals $cot(\phi\ /2)\, e^{i\theta}$. The multiplier $e^{i\theta}$ did not figure into the calculation of the elapsed time, because the cross section in the plot is taken at the angle θ. Perhaps it would be more accurate to say that stereographic projection can be used to calculate the horizontal component of a photon, the elapsed time when the height above the $\mathbb{C}$-plane equals the distance traversed by F, the length of F, and the height of F above the plane.

One can also algebraically deduce that the horizontal component of F equals $\cot(\phi/2)$:

Triangles pS $\tilde{z}$ and N0z share the angle at N. They have angle $\phi/2$ at S and z. They both have right triangles by construction. Hence, they are similar.

$$\sin(\phi/2) = \frac{1}{|p-z|}$$
$$\cos(\phi/2) = \frac{z}{|p-z|}$$
$$\cot(\phi/2) = \frac{z}{|p-z|} \cdot \left| p - z \right| = z$$

$$R = \frac{\cot(\phi/2)}{\sin\phi} \quad \text{(from above)}$$

$$H_F = R \sin\phi = \frac{\cot(\phi/2)}{\sin\phi} \sin\phi = \cot(\phi/2)$$

Taking the θ angle into account, one can deduce *that F hits* $\mathbb{C}$ *at the point* $z = cot(\phi/2)\, e^{i\theta}$. *Thus, Penrose's construction is equivalent to stereographic projection.* But I think one should bear in mind that the "$\mathbb{C}$" that is being "hit" is a copy of the complex plane elevated at the speed of light for the same amount of time as the travel time of the photon that strikes it.