Needham, VDGF, Chapter 7, Exercise 32, p. 92.

G. Palmer, September 30, 2022, 10:40 am PDT

32. Horosphere Metric. Let us derive the result of Exercise 30 in another way. Define a coordinate r such that

$$dr = \frac{d\phi}{1+\cos\phi}$$

Show that the metric of the horosphere (see Ex. 30) can then be written as $d\hat{s}^2 = dr^2 + r^2\, d\theta^2$*, which is just the metric of the Euclidean plane in polar coordinates.*

Solve for r.

$$\int dr = \int \frac{1}{1+\cos\phi}\, d\phi$$

$$r = \frac{\sin\phi}{1+\cos\phi} + \text{const.} \qquad (1)$$

Solve for $d\phi$.

$$d\phi = (1 + \cos\phi)\, dr \qquad (2)$$

Substitute both results (1) and (2) into result of Ex. 30. Let the constant of integration in (1) equal 0.

$$d\hat{s}^2 = \frac{\sin^2\phi\, d\theta^2 + d\phi^2}{(1+\cos\phi)^2} = \frac{\sin^2\phi\, d\theta^2 + ((1+\cos\phi)\, dr)^2}{(1+\cos\phi)^2}$$

$$= r^2\, d\theta^2 + dr^2$$

Rearranged, this is what we wanted.