Needham, VDGF, Chapter 7, Exercise 31, p. 92.
G. Palmer, November 3, 2022, 9:40 am PDT

Out[589]=

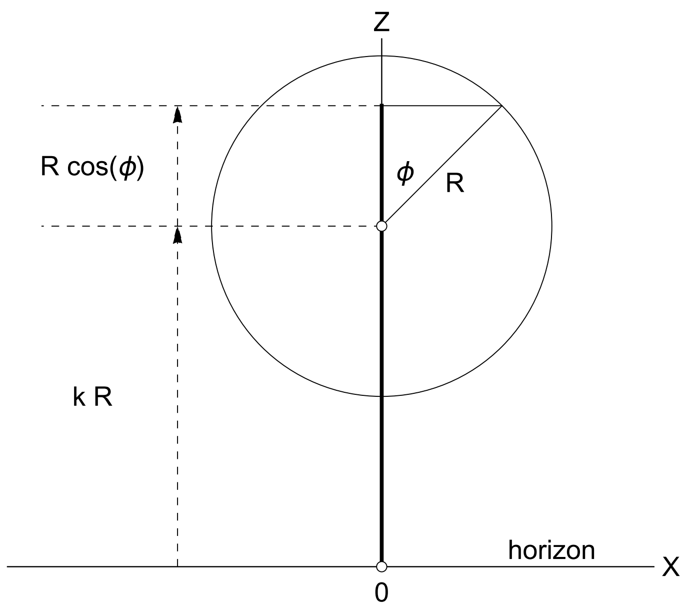

Figure 1. Spheres in $\mathbb{H}^3$
Cross–section through X and Z axes

31. Spheres in $\mathbb{H}^3$. Consider a Euclidean sphere of radius R centred at height kR above the horizon of $\mathbb{H}^3$. Using the derivation of (6.24) as your inspiration, show that the metric of such a sphere is

$$d\hat{s}^2 = \frac{\sin^2\phi \, d\theta^2 + d\phi^2}{(k+\cos\phi)^2}.$$

Referring to figure 1, let $Z = R(k + \cos\phi)$. Then

$$d\hat{s} = \frac{\mathrm{ds}}{\text{Euclidean height above boundary}}$$

$$= \frac{R\sqrt{\sin^2\phi \, \mathrm{d}\theta + \mathrm{d}\phi^2}}{R(k+\cos\phi)} = \frac{\sqrt{\sin^2\phi \, \mathrm{d}\theta + \mathrm{d}\phi^2}}{k+\cos\phi}$$

$$\Longrightarrow d\hat{s}^2 = \frac{\sin^2\phi \, \mathrm{d}\theta^2 + \mathrm{d}\phi^2}{(k+\cos\phi)^2}$$

Apply the curvature formula (4.10) to confirm that if this sphere lies entirely above the horizon (i.e. k > 1), then intrinsically this surface is genuinely spherical, with constant curvature, $\mathcal{K} = k^2 - 1$. ...

$$B = \frac{1}{k+\cos\phi}, A = \frac{\sin\phi}{k+\cos\phi} = B\sin\phi$$

$$\partial_\phi(A) = -\frac{1+k\cos\phi}{(k+\cos\phi)^2} = (1+k\cos\phi)\,B^2$$

$$\partial_\phi(A)\big/B = \ (1+k\cos\phi)\,B$$

$$\partial_\phi\left[\partial_\phi(A)\big/B\right] = \frac{\left(k^2-1\right)\sin\phi}{(k+\cos\phi)^2} = -\left(k^2-1\right)\mathrm{AB}$$

$$\partial_\theta(B)/A = 0, \qquad \partial_\theta(\partial_\theta(B)/A) = 0$$

$$\mathcal{K} = -\frac{1}{\mathrm{AB}}\left[\left(k^2-1\right)(-\mathrm{AB}) - 0\right] = \left(k^2-1\right)$$