Needham, VDGF, Chapter 7, Exercise 30, p. 92.
G. Palmer, November 2, 2022, 1 pm PDT

Out[1015]=

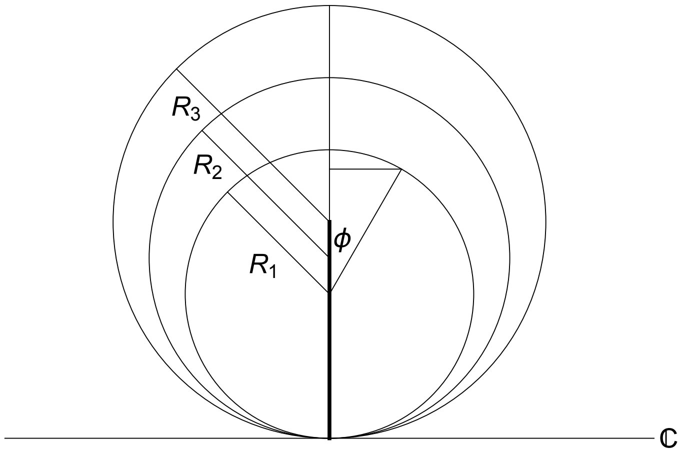

Figure 1. Horospheres
Thick line is Z = $R_i(1+\cos\phi)$
cross–section on X and Z Axes

30. Horosphere Metric and Curvature. As discussed, a typical horosphere of $\mathbb{H}^3$ appears on a sphere resting on $\mathbb{C}$. Using the derivation of (6.24) as your inspiration, show that the metric of such a horosphere is

$$d\hat{s}^2 = \frac{\sin^2\theta\, d\theta^2 + d\phi^2}{(1+\cos\phi)^2}.$$

Let $Z = R(1+\cos\phi)$. Then, from (6.23),

$$d\hat{s} = \frac{\text{ds}}{\text{Euclidean height above boundary}}$$

$$= \frac{R\sqrt{\sin^2\phi\, d\theta + d\phi^2}}{R(1+\cos\phi)} = \frac{\sqrt{\sin^2\phi\, d\theta + d\phi^2}}{1+\cos\phi}$$

$$\Longrightarrow d\hat{s}^2 = \frac{\sin^2\phi\, d\theta^2 + d\phi^2}{(1+\cos\phi)^2}$$

Apply the curvature formula (4.10) to confirm that the horosphere is intrinsically a flat Euclidean plane: $\mathcal{K} = 0$.

$$B = \frac{1}{1+\cos\phi}, A = \frac{\sin\phi}{1+\cos\phi} = B\sin\phi$$

A = B $\sin\phi$

$$\partial_\phi(A) = \frac{\cos\phi}{1+\cos\phi} + \frac{\sin^2\phi}{(1+\cos\phi)^2} = \frac{1}{1+\cos\phi} = B$$

$$\partial_\phi(A)\big/B = 1$$

$$\partial_\phi\left[\partial_\phi(A)\big/B\right] = \partial_\phi[1] = 0$$

$$\partial_\theta(B)/A = 0, \quad \partial_\theta(\partial_\theta(B)/A) = 0$$

$$\mathcal{K} = -\frac{1}{\mathrm{AB}}[0-0] = 0$$