Needham, VDGAF, Chapter 7, Exercise 3, p. 83
Palmer, February 5, 2022, 7:50 am PST

3. Central Projection. Rederive the central projection metric (4.2) by calculation.

This is the same as the derivation in my notes for Chapter 4.

By inspection of 4.1, we have $s_{\text{lat}} = r\theta$ on the plane, so we can write an equation for an infinitesimal arc ds_1 on the plane corresponding to an infinitesimal line of latitude $d\hat{s}_1$ on the sphere.

$$ds_1 = r\,d\theta$$

$$d\hat{s}_1 = \frac{R}{H}\,ds_1 = \frac{R}{\sqrt{R^2+r^2}}\,r\,d\theta$$

A similar equation for an arc emanating from z corresponds to a line of longitude of the sphere.

$$s_{\text{lon}} = R\phi$$

Since $\phi = \tan^{-1}(r/R)$,

$$d\phi = \frac{1}{\left(1+\frac{r^2}{R^2}\right)R}\,dr$$

$$d\hat{s}_2 = R\,d\phi = \frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,dr$$

Then the metric on the surface is

$$(d\hat{s}_2)^2 = \left(\frac{R}{\sqrt{R^2+r^2}}\,r\,d\theta\right)^2 + \left(\frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,dr\right)^2$$

$$= \frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,r^2\,d\theta^2 + \frac{1}{\left(1+\frac{r^2}{R^2}\right)^2}\,dr^2$$

$$= \frac{1}{\left(1+\frac{r^2}{R^2}\right)}\left[\frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,dr^2 + r^2\,d\theta^2\right]$$

This is (4.2)