Needham, VDGF, Chapter 7, Exercise 28, pp. 91-92.
Palmer, September 26, 2022, 20 am PDT

28. Metric of $\mathbb{H}^2$ within $\mathbb{H}^3$. Let us explore different ways of expressing the metric (6.24) of a typical hyperbolic planes (sic) in $\mathbb{H}^3$, namely, a horizon-centred hemisphere.

(i) Show that if we define $u = \tan\phi$, the metric (6.24) becomes

$$d\hat{s}^2 = u^2\,\mathrm{d}\theta^2 + \frac{\mathrm{du}^2}{1+u^2}$$

and use (4.10) to confirm that $\mathcal{K} = -1$.

Metric (6.24) is

$$d\hat{s}^2 = \frac{\sin^2\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2}{\cos^2\phi}$$

The previous usage of ϕ on p. 48 [4.12][b] shows it as the angle about the origin in any direction from the Z axis. That information is not needed to complete this problem.

$$\mathrm{du} = \sec^2\phi\,\mathrm{d}\phi$$

$$\mathrm{d}\phi = \frac{\mathrm{du}}{\sec^2\phi} = \frac{\mathrm{du}}{1+\tan^2\phi}$$

$$\mathrm{d}\phi^2 = \frac{\mathrm{du}^2}{\left(1+\tan^2\phi\right)^2} = \frac{\mathrm{du}^2}{\left(1+\tan^2\phi\right)\times\left(1+u^2\right)}$$

$$d\hat{s}^2 = \frac{\sin^2\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2}{\cos^2\phi} = \frac{\sin^2\phi}{\cos^2\phi}\,\mathrm{d}\theta^2 + \frac{1}{\cos^2\phi}\,\mathrm{d}\phi^2$$

$$= \tan^2\phi\,\mathrm{d}\theta^2 + \frac{1}{\cos^2\phi}\,\frac{\mathrm{du}^2}{\left(1+\tan^2\phi\right)\times\left(1+u^2\right)}$$

$$= u^2\,\mathrm{d}\theta^2 + \frac{\mathrm{du}^2}{1+u^2} \qquad (1)$$

This is what we wanted. As Vasco has pointed out (P.C., VDGF), this is a mapping from the sphere to the complex plane.

and use (4.10) to confirm that $\mathcal{K} = -1$.

(1) may be used to calculate $\mathcal{K}$ using (4.10).

$$\text{A} = \text{u}; \qquad B = \left(1/\left(1+u^2\right)\right)^{\frac{1}{2}}$$

$$\partial_u(A)/B = 1/B = B^{-1}$$

$$\partial_u\left(B^{-1}\right) = \frac{u}{\sqrt{1+u^2}} = \text{AB}$$

$$\partial_\theta(B)/A = 0 \Longrightarrow \partial_u(\partial_u(B)/A) = 0$$

$$\mathcal{K} = -\frac{1}{\text{AB}}[\text{AB}] = -1$$

(ii) Show that if we define a new variable ξ via u ≡ 1/sinh ξ, the metric of the previous part becomes conformal:

$$d\hat{s}^2 = \frac{\mathrm{d}\theta^2 + \mathrm{d}\xi^2}{\sinh^2\xi}$$

$$\text{du} = d(1/\sinh\xi) = -\coth\xi\,\text{csch}\,\xi\,\mathrm{d}\xi$$

Begin with

$$d\hat{s}^2 = u^2\,\mathrm{d}\theta^2 + \frac{\mathrm{du}^2}{1+u^2} = \left(1/\sinh^2\xi\right)\mathrm{d}\theta^2 + \frac{(-\coth\xi\,\text{csch}\,\xi\,\mathrm{d}\xi)^2}{1+1/\sinh^2\xi}$$

Just look at the second term:

$$\frac{(-\coth\xi\,\text{csch}\,\xi\,\mathrm{d}\xi)^2}{1+1/\sinh^2\xi} = \frac{\coth^2\xi\,\text{csch}^2\xi\,\mathrm{d}\xi^2}{\left(\sinh^2\xi+1\right)/\sinh^2\xi\right)} = \frac{\sinh^2\xi\coth^2\xi\,\text{csch}^2\xi\,\mathrm{d}\xi^2}{\sinh^2\xi+1}$$

$$= \frac{\sinh^2\xi\left(\cosh^2\xi/\sinh^2\xi\right)\left(1/\sinh^2\xi\right)\mathrm{d}\xi^2}{\cosh^2\xi} = \frac{\mathrm{d}\xi^2}{\sinh^2\xi}$$

Now one can write

$$d\hat{s}^2 = \frac{\mathrm{d}\theta^2 + \mathrm{d}\xi^2}{\sinh^2\xi}$$

Since $\mathrm{ds}^2 = \mathrm{d}\theta^2 + \mathrm{d}\xi^2$, $\Lambda^2 = 1/\sinh^2\xi$. The equation conforms to the metric (4.14), p. 40, with conformal coordinates and to the conformal map (4.13), p. 39,

$$d\hat{s} = \Lambda(z)\, \mathrm{ds},$$

where $\Lambda = (1/\sinh\xi)$ and $\mathrm{ds} = \sqrt{\mathrm{d}\theta^2 + \mathrm{d}\xi^2}$.

(iii) Finally, find a conformal mapping of the (θ, ξ)*-plane to the (x,y)-plane such that the conformal hyperbolic metric in part (ii) takes the standard form (5.7)*

$$d\hat{s}^2 = \frac{\mathrm{dx}^2 + \mathrm{dy}^2}{y^2}$$

(Hint: Set dy / y = $d\xi / \sinh\xi$ *.)*

By the hint, in the result of (ii), one can substitute dy/y for $\mathrm{d}\xi/\sinh\xi$.

$$d\hat{s}^2 = \frac{\mathrm{d}\theta^2}{\sinh^2\xi} + \frac{\mathrm{dy}^2}{y^2}$$

Then, since the metric must remain conformal, A = B, and since y^2 is substituted for $\sinh^2\xi$, one can substitute again and write

$$d\hat{s}^2 = \frac{\mathrm{d}\theta^2}{y^2} + \frac{\mathrm{dy}^2}{y^2}$$

which is rewritable in the format of (4.14), p. 40,

$$d\hat{s}^2 = \frac{1}{y^2}\left(\mathrm{d}\theta^2 + \mathrm{dy}^2\right)$$

Then if it is to take the standard form (5.7), it must be the case that $\mathrm{dx} = \mathrm{d}\theta$, so one can write

$$d\hat{s}^2 = \frac{\mathrm{dx}^2 + \mathrm{dy}^2}{y^2}$$

To write the mapping, one must write $\mathrm{d}\theta$ and $\mathrm{d}\xi$ in terms of dx and dy. Then, one can substitute the axis mappings into

$$d\hat{s}^2 = \frac{\mathrm{d}\theta^2 + \mathrm{d}\xi^2}{\sinh^2\xi} \qquad (1)$$

We want to find

$$\mathrm{d}\theta = \partial_x \theta\, \mathrm{dx} + \partial_y \theta\, \mathrm{dy}$$

$$d\xi = \partial_x \xi dx + \partial_y \xi dy$$

By the hint, $dy/y = d\xi/\sinh\xi$, so $\partial_y \xi = \sinh\xi/y$. Then, because the expansion factor must be the same for θ and ξ, let $\partial_x \theta = \sinh\xi/y$. Note that θ is independent of y and ξ is independent of x. Hence, $\partial_y \theta = 0$ and $\partial_x \xi = 0$. Now we can write both mappings:

$$d\theta = \partial_x \theta dx + 0 = \frac{\sinh\xi}{y} dx \qquad (2)$$

$$d\xi = 0 + \partial_y \xi dy = \frac{\sinh\xi}{y} dy \qquad (3)$$

It appears that we are only partially successful in writing $d\theta$ entirely in terms of x and y in that the RHS's of (2) and (3) contain $\sinh\xi$. In writing the metric, this incompleteness in the mapping is cancelled out by $\sinh^2 \xi$ in the denominator of (1).

Substitute from (2) and (3) into $d\hat{s}^2 = \frac{d\theta^2 + d\xi^2}{\sinh^2 \xi}$.

$$d\hat{s}^2 = \frac{\left(\frac{\sinh\xi}{y} dx\right)^2 + \left(\frac{\sinh\xi}{y} dy\right)^2}{\sinh^2 \xi} = \frac{dx^2 + dy^2}{y^2}$$

which we already know has the conformal format of (4.14). Vasco used the Cauchy-Riemann test to verify that the mapping is conformal.

It would help if one had a diagram to illustrate the meaning of x, y, θ, and ξ. It may be that models in [5.4] and [5.5] would serve to define x, y, and $\theta = x$, but then what is ξ? Vasco looks at the mapping as "just changes in the coordinates of the flat map." (P.c., VDGF) Looking at it that way is helpful, but one would also like to know why it works out so neatly.