Needham, VDGF, Chapter 27, Parts (viii, ix), p. 91.
G. Palmer, August 24, 2022, 2:15 pm PDT

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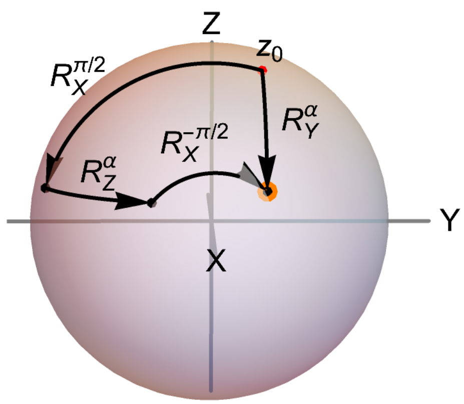

Figure 3. $[R_Y^{\alpha}]=[R_X^{-\pi/2}][R_Z^{\alpha}][R_X^{\pi/2}]$

(viii) Explain geometrically why for any angle α,

$$[R_Y^{\alpha}] = \left[R_X^{-(\pi/2)}\right][R_Z^{\alpha}]\left[R_X^{(\pi/2)}\right]$$

The rotations indicated in the equation were plotted directly in Mathematica. They are all rotations of the sphere. “R rotates z” means that the rotation of S induces rotation of z.

$[R_Y^{\alpha}]$ rotates z anticlockwise by α radians about the Y axis.

$\left[R_X^{(\pi/2)}\right]$ rotates z anticlockwise by $\pi/2$ radians about the X axis in a plane parallel to the YZ plane.
$[R_Z^{\alpha}]$ rotates z anticlockwise by α radians about the Z axis in a plane parallel to the XY plane.
$\left[R_X^{-(\pi/2)}\right]$ rotates z about the X axis clockwise, reversing the first rotation in the compound matrix.

The rotations $\left[R_X^{-(\pi/2)}\right]$ and $\left[R_X^{(\pi/2)}\right]$ occur in parallel planes as shown in Figure 4. The angles of rotation are of equal magnitude but opposite in direction (Figures 3 and 4). If the planes were conflated, the circles of rotation would be concentric. $[R_Z^{\alpha}]$ takes the terminal point of $\left[R_X^{(\pi/2)}\right]$ to the initial point of $\left[R_X^{-(\pi/2)}\right]$ in the plane XY. All the points and curves of rotation can be plotted on two planes parallel to the YZ plane and two planes parallel to the XY and XZ planes.

Out[ ]=

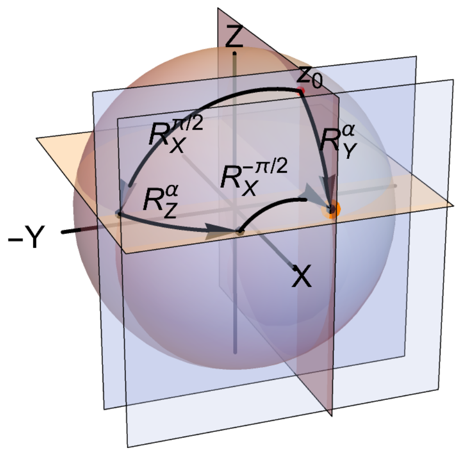

Figure 4. Axes and planes of rotation

(ix) Finally, consider a general rotation about an arbitrary axis A. By rotating A into the (X, Y)-plane and then rotating it to the Y-axis, deduce that this too can be represented as a matrix in SU(2), thereby concluding the proof.

Let $\mathbb{A}$ be an axis in the XY plane orthogonal to the projection $\tilde{A}$ of A onto the XY plane. Then, the rotation of A into the XY plane can be written $\left[R_{\mathbb{A}}^{\phi}\right]$. This rotation should move any point on the sphere to a point on the equator. It should have the form $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$. The rotation rotating $\tilde{A}$ to the Y-axis can be written $\left[R_Z^{\theta}\right]$. This should also have the form $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$. A general rotation about Y can be written $[R_Y^{\alpha}]$. This also has the form $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$. Again, it is understood that the rotation occurs in the surface of the sphere. It is, in fact, a rotation *of* the sphere. Then $\left[R_Z^{-\theta}\right]$ rotates the axis back to its original theta angle and $\left[R_{\mathbb{A}}^{-\phi}\right]$ rotates the axis back to A.

$$[R_A^{\alpha}] = \left[R_{\mathbb{A}}^{-\phi}\right]\left[R_Z^{-\theta}\right][R_Y^{\alpha}]\left[R_Z^{\theta}\right]\left[R_{\mathbb{A}}^{\phi}\right]$$

Notice that the axes of $[R_A^{\alpha}]$ and $\left[R_{\mathbb{A}}^{-\phi}\right]$ are orthogonal. This orthogonality of the axes in the result (LHS) compared to the axis of the first and last transformation (RHS) is also true of the transformation in exercise (viii).

Since all of the composite matrices on the RHS belong to the spherical geometry group, which is a special unitary group, their composition $[R_A^{\alpha}]$ also belongs to *SU(2)*. We know that the 2 by 2 spherical geometry group is special (normalized with unit determinant), because $|a| + |b| = 1$. We know that it is unitary because it has a conjugate transpose (inverse) that takes it to the identity matrix.