Out[46]=

Needham, VDGF, Chapter 7 (iv-vii), p. 91.
G. Palmer, August 24, 11 am PDT

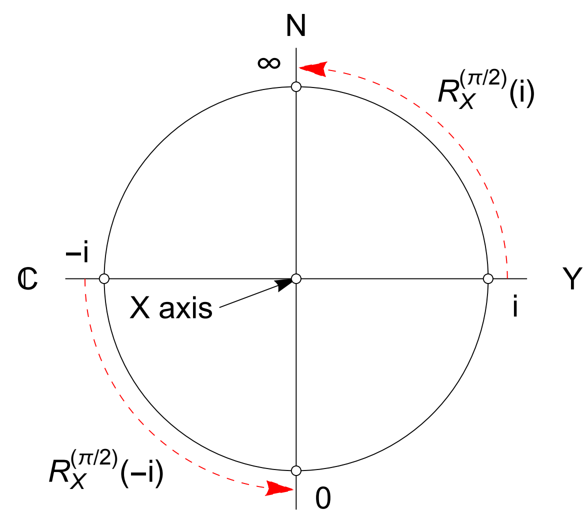

Figure 2. Rotation $R_X^{(\pi/2)}(z)$

(iv) To confirm that $R_X^{(\pi/2)}(z)$ is in fact a Möbius transformation, let us suppose for now that it is, and let us try to guess its form. Use a sketch to confirm that $-i \to 0$, so the numerator must be proportional to (z+i). Likewise, by finding the point that rotates to the north pole, deduce the denominator. This determines the form of $R_X^{(\pi/2)}(z)$ up to a multiplicative factor; determine this factor by noting that $0 \to i$. Conclude that if $R_X^{(\pi/2)}(z)$ is a Möbius transformation, it can only be

$$R_X^{(\pi/2)}(z) = \frac{z+i}{iz+1}$$

Deduce the denominator: The point that rotates to N is *i*. The denominator must sum to 0. I'm not sure at this point how one would determine whether to let d = ±1 or ±i, so some trial and error is pursued. If we let d = 1, then ci must be −1, so c must be *i*. Then we have

$$\frac{z+i}{iz+1}$$

which is the desired transformation. But if we let $d = i$, then ci must be $-i$, so c must be −1 and we obtain

$$\frac{z+i}{-z+i}.$$

In this case also, i rotates to N. In the first case, $0 \to i$. In the second case $0 \to 1$, so only the first is valid. It could be that $d = -1$ or $-i$. Then the denominator could be −iz − 1 giving

$$\frac{z+i}{-iz-1},$$

but this is only (-1)$\frac{z+i}{iz+1}$, so it is essentially the same as the given formula, or the denominator could be z-i, giving

$$\frac{z+i}{z-i},$$

for which $0 \to -1$ rather than the required i.

Since only multiplying both top and bottom by the same complex number preserves the mapping, one can then conclude that *if $R_X^{(\pi/2)}(z)$ is a Möbius transformation, it can only be*

$$R_X^{(\pi/2)}(z) = \frac{z+i}{iz+1}.$$

(v) Show that this is in fact $R_X^{(\pi/2)}(z)$ by demonstrating that it satisfies the equation in part (iii). (Hints: Multiply top and bottom by (X-iZ), but only multiply through in the denominator. Finally, in the resulting denominator, use the fact that $X^2 + Y^2 + Z^2 = 1$.

The equation in part (iii) was given as $z = \frac{X+iY}{1-Z} \longrightarrow R_X^{(\pi/2)}(z) = \frac{X-iZ}{1-Y}$. In working part (iii), it was discovered that the expression "$R_X^{(\pi/2)}(z)$" implies a tacit projection of z to the sphere and a projection back from the sphere to the plane which I have labeled $\tilde{z}$. These two points project as (Z + iY) and (Y – iZ) respectively. The rotation was applied to (Z + iY) and then projected back from the result (Y – iZ) to $\frac{X-iZ}{1-Y}$. But in this part (v) of the exercise, $R_X^{(\pi/2)}(z)$, given as a Möbius transformation, is applied directly to points on the complex plane. These points do not need to be on the sphere. There is no explicit projection from $\mathbb{C}$ to S and back. The transformation $z \to \tilde{z}$ is from $\mathbb{C}$ to $\mathbb{C}$, where it has the appearance of a rotation with an expansion/contraction. (See Figure 1, (i-iii)). The proof of the equation in (iii) is given through a bit of algebra from Vasco's answer. Substitute the first point $\frac{X+iY}{1-Z}$ for z in the equation from (iv) and multiply top and bottom by (X-iZ) according to the first hint.

$$R_X^{(\pi/2)}(z) = \frac{z+i}{iz+1} = \frac{\frac{X+iY}{1-Z}+i}{i\frac{X+iY}{1-Z}+1} = \frac{X+iY+i(1-Z)}{i(X+iY)+(1-Z)} \frac{(X-iZ)}{(X-iZ)}$$

In the denominator, $i(X$ + iY) + (1 – Z) can be rewritten (i(X+iZ) + (1-Y). This is a key step, because it permits one to multiply the conjugates (X+iZ)(X-iZ) to obtain $X^2 + Z^2$, for which one can substitute $\left(1 - Y^2\right)$ according to the second hint. This is factored into $(1 + Y) \times (1 - Y)$ enabling one to separate out (1-Y) as a common factor and cancel the left hand term in the numerator.

Multiply conjugates in the denominator.

$$= \frac{[X+iY+i(1-Z)](X-iZ)}{i\,\mathbf{(X+iZ)\,(X-iZ)}+(1-Y)(X-iZ)} = \frac{[X+iY+i(1-Z)](X-iZ)}{i\,\mathbf{(X^2+Z^2)}+(1-Y)(X-iZ)}$$

Substitute $(1-Y^2)$ for (X^2+Z^2) and factor out (1-Y).

$$= \frac{[X+iY+i(1-Z)](X-iZ)}{i\,\mathbf{(1-Y^2)}+(1-Y)(X-iZ)} = \frac{[X+iY+i(1-Z)](X-iZ)}{i(\mathbf{(1-Y)}\times(1+Y))+(1-Y)(X-iZ)} = \frac{[X+iY+i(1-Z)](X-iZ)}{\mathbf{(1-Y)}(i(1+Y)+(X-iZ))}$$

Cancel equal factors in numerator and denominator.

$$= \frac{\mathbf{[X+iY+i(1-Z)]}(X-iZ)}{(1-Y)\,\mathbf{[X+iY+i(1-Z)]}} = \frac{X-iZ}{1-Y}$$

This is what we wanted.

(vi) Deduce that $R_X^{(\pi/2)}$ can be represented in SU(2) as

$$\left[R_X^{(\pi/2)}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Begin with $\left[R_X^{(\pi/2)}\right]$ from (iv). The multiplier $\pm\frac{1}{\sqrt{\text{ad}-\text{bc}}} = \pm\frac{1}{\sqrt{2}}$ normalizes $\left[R_X^{(\pi/2)}\right]$. Hence,

$\left[R_X^{(\pi/2)}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ is one of the two possible normalized matrices associated with the transformation.

(vii) Explain geometrically why $\left[R_X^{(\pi/2)}\right]^2 = [R_X^{\pi}]$, and confirm this by direct calculation.

Geometrically, squaring a complex number doubles the angle about the origin. Since $\left[R_X^{(\pi/2)}\right]$ effects a rotation of a complex number by $\pi/2$ about the X axis, $\left[R_X^{(\pi/2)}\right]^2$ effects a rotation of π about the X axis. Write $[z]$ as $\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^2 \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} i\,\zeta_2 \\ i\,\zeta_1 \end{bmatrix}$$

The swapping of the homogeneous coordinates between numerator and denominator corresponds to a rotation of $\pi/2$ about X. Multiplication of the resulting column vector by *i* corresponds to another rotation of $\pi/2$.

Confirm by direct calculation that $\left[R_X^{(\pi/2)}\right]^2 = [R_X^{\pi}]$. If the rotation of the *sphere* (seen from the inside) under $\left[R_X^{(\pi/2)}\right]$ is $(-\pi/2)$, then the rotation under $\left[R_X^{(\pi/2)}\right]^2$ is $(-\pi)$.

Rotation of π sends i to $-i$. Write $i \equiv \begin{bmatrix} i \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} \equiv -i$$

...($-i$) to i. Write $-i \equiv \begin{bmatrix} -i \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} \equiv i$$

...∞ (N) to 0 (S). Write $\infty \equiv \begin{bmatrix} \zeta \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{i}\zeta \end{bmatrix} \equiv 0$$

... 0 to ∞. Write $0 = \begin{bmatrix} 0 \\ \zeta \end{bmatrix}$

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \zeta \end{bmatrix} = \begin{bmatrix} \mathrm{i}\zeta \\ 0 \end{bmatrix} \equiv \infty$$

These are all rotations of π or ($-\pi$) in the XZ plane.