Needham, Chapter 7, Exercise 27, parts (i-iii), pp. 90-91.
G. Palmer, August 24, 2022, 2:45 pm PDT

27. Moebius Rotations of the Riemann Sphere. ... Let RθZ represent a clockwise rotation of the Riemann Sphere through an angle θ about the positive Z-axis of (X,Y,Z)-space, so that RθZ(z)=e^{iθ}(z).

Because notebooks produced in the Mathematica editor are easily corrupted by complicated notations and formats, rotation by angle θ about Z is written as RθZ. Expressions with e superscripted are written e^{iθ}(z), for example, and round brackets are used instead of the square brackets to indicate Moebius matrices with complex numbers.

In the introductory paragraph it is declared that *RθZ(z) = e^{iθ}(z)* represents a clockwise rotation. The exercise seems more straightforward if one adopts the position that positive θ is an anticlockwise rotation.

(i) Find a matrix in [RθZ] in SU(2) representing RθZ.

Write *e^{iθ}(z)* as a Moebius transformation and then as a matrix of the transformation.

$$e\text{^}\{i\theta\}(z) = \frac{ze\text{^}\{i\theta\}+0}{0+1} \equiv \begin{pmatrix} e\text{^}\{i\theta\} & 0 \\ 1 & 0 \end{pmatrix}$$

Normalize by multiplying by $\pm 1 \big/ \sqrt{\text{ad} - \text{bc}} = \pm e\text{^}\{-i\theta/2\}$.

$$\pm e\text{^}\{-i\theta/2\} \begin{pmatrix} e\text{^}\{i\theta\} & 0 \\ 0 & 1 \end{pmatrix} = \pm \begin{pmatrix} e\text{^}\{i\theta/2\} & 0 \\ 0 & e\text{^}\{-i\theta/2\} \end{pmatrix}$$

This has the conjugate transpose (inverse) $\pm \begin{pmatrix} e\text{^}\{-i\theta/2\} & 0 \\ 0 & e\text{^}\{i\theta/2\} \end{pmatrix}$.

RθZ is normalized with determinant 1 and it goes to identity under the conjugate transpose, so it is unitary . It is therefore a member of SU(2).

(ii) Explain why the result (6.6) can be written

$$[\text{R}\theta\text{Z}] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \text{SU}(2)$$

6.6: Complex inversion, $z \to (1/z)$, is a rotation of the Riemann sphere, through an angle π about the real axis. It is therefore conformal and maps circles to circles.

Rewrite (1/z) as a Moebius matrix:

$$(1/z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Coefficients of Moebius transformations are not unique. Any matrix may be multiplied by an arbitrary complex number without changing the mapping (*VCA*, pp. 149-150). Therefore, one can write

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \text{for} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

without changing the mapping.

The conjugate transpose is $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{identity}$$

The matrix of complex inversion is normalized with determinant 1 and it has a conjugate transpose (inverse). It is therefore a member of SU(2).

Out[◦]=

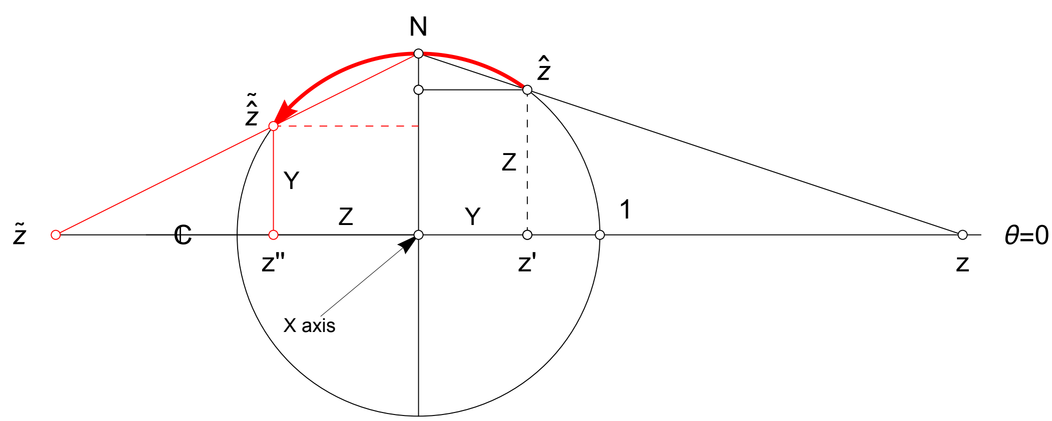

Figure 1. Rotation $R_X^{\pi/2}(\hat{z})$ on Riemann sphere
in YZ–plane orthogonal to X axis
with projections to ℂ

(iii) Referring to (4.24), use a sketch to show that the complex mapping corresponding to $R(\pi/2)\,X$ is

$$z = \frac{X+iY}{1-Z} \to R(\pi/2)\,X(z) = \frac{X-iZ}{1-Y}$$

See Figure 1. z and $\tilde{z}$ (tilde z) are plotted as $\frac{X+\mathrm{i}Y}{1-Z}$ and $\frac{X-\mathrm{i}Z}{1-Y}$ respectively. The projection of z to the sphere is rotated $\pi/2$ about the X-axis. The resulting point is projected back to the complex plane. After rotation, $X+\mathrm{i}Y$, labeled z', becomes $X-\mathrm{i}Z$, labeled z''. Of the two small triangles touching at N, the height $(1-Z)$ of the small triangle on the right, is transformed by the $\pi/2$ rotation to the height $(1-Y)$ of the larger one on the left. Now, referring to p. 48, one can see that if

$$\frac{|z|}{|z'|} = \frac{1}{1-Z}, \text{ then } \frac{|\tilde{z}|}{|z''|} = \frac{1}{1-Y}$$

and that by similar triangles to the left of the Z axis, and similar triangles on the RHS*,

$$z = \frac{X+\mathrm{i}Y}{1-Z} \to R(\pi/2)\,X(z) = \frac{X-\mathrm{i}Z}{1-Y}.$$

*The similar triangles on the LHS are not similar to those on the RHS, as they have only one angle in common, the right angle at the Z-axis.