Needham, VDGF, Chapter 7, Exercise 26, p. 90.
G. Palmer, October 29, 2022, 3:30 am PDT

26. Curvature of the Beltrami-Poincaré disc. *Use the conformal curvature formula (4.16) to verify that the Beltrami-Poincaré disc with metric (5.15) has constant negative curvature $\mathcal{K} = -1$.*

On p. 62, ¶ -1, we find *You can at least confirm that this* [(5.15)] *is indeed* $\mathbb{H}^2$ *by using the conformal curvature formula (4.16) to verify [exercise] that this surface has constant negative curvature* $\mathcal{K} = -1$. The conformal curvature formula is:

$$\mathcal{K} = -\frac{\nabla^2 \ln \Lambda}{\Lambda^2}.$$

Metric (5.15), p. 62, is

$$d\hat{s} = \frac{2}{1-r^2}\,\mathrm{ds}$$

From (5.15) and ¶ 2, p. 41 one finds that $\Lambda = 2/(1-r^2)$. Calculate $\nabla^2 \ln \Lambda = \partial_r \partial_r \ln \Lambda$.

$$\partial_r \partial_r \ln \Lambda = \Lambda + r \cdot \mathrm{r}\Lambda^2 \quad \text{(see below for calculation)}$$

This Cartesian form of the Laplacian turns out to be insufficient as will be explained.

$$\mathcal{K} = -\frac{\nabla^2 \ln \Lambda}{\Lambda^2} = -\frac{\partial_r \partial_r \ln \Lambda}{\Lambda^2} = -\frac{\Lambda + r \cdot \mathrm{r}\Lambda^2}{\Lambda^2} = -\frac{1 + r^2 \Lambda}{\Lambda}$$

$$= -\left(\frac{1-r^2}{2} + r^2\right) = -\left(\frac{1-r^2}{2} + \frac{2r^2}{2}\right) = -\frac{1+r^2}{2}$$

This is not what we wanted. I thought the two dimensions (θ,r) would serve as well as (x, y), for this exercise, but it appears that they do not.

Vasco (p.c. VDGF Forum) pointed out that the polar form of the Laplacian would be more appropriate than the Cartesian form which I have used. The polar form (4.23), p. 47, is

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\partial^2{}_\theta$$

Since θ is not a part of the metric, this can be written

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\,\partial_r$$

There seems to be no further instruction on finding Λ, so we assume that can be found in the same way: as the coefficient of ds in the metric (p. 41, ¶ 2), so we let $\Lambda = \frac{2}{(1-r^2)}$.

The Laplacian applied to Λ becomes

$$\nabla^2 = \partial_r^2 \ln\Lambda + \frac{1}{r}\,\partial_r \ln\Lambda$$

$$\partial_r \Lambda = \frac{4r}{\left(1-r^2\right)^2} = \mathrm{r}\Lambda^2$$

$$\partial_r \ln\Lambda = \frac{1}{\Lambda}\,\partial_r \Lambda = \frac{\mathrm{r}\Lambda^2}{\Lambda} = \mathrm{r}\Lambda$$

$$\partial_r\left(\partial_r \ln\Lambda\right) = \partial_r\left(\mathrm{r}\Lambda\right) = \Lambda + r\,\partial_r\Lambda = \Lambda + r\cdot\mathrm{r}\Lambda^2$$

$$\nabla^2 \ln\Lambda = \left(r^2\Lambda^2 + \Lambda\right) + \frac{1}{r}\,r\,\Lambda = r^2\Lambda^2 + 2\Lambda$$

$$\mathcal{K} = -\frac{r^2\Lambda^2 + 2\Lambda}{\Lambda^2} = -\frac{r^2\Lambda + 2}{\Lambda} = -\left(r^2 + \frac{2}{\Lambda}\right)$$

$$= -\left(r^2 + \left(1 - r^2\right)\right) = -1$$

This is what we wanted. Notice that the polar form of the Laplacian increases $\left(\nabla^2 \ln\Lambda\right)$ by one Λ over the Cartesian form.