Needham, VDGF, Chapter 7, Exercise 25, p. 90.
G. Palmer, October 29, 2022, 10:15 PDT

25. Metric of the Beltrami-Poincaré Disc. *We know from [6.3]* [Perhaps he means [6.4] and ¶ 3 underneath it.] *that Möbius transformations can map half-planes to discs, and, ...it turns out that a special Möbius transformation from the upper half-plane model of* $\mathbb{H}^2$ *[5.12] to the disc model of* $\mathbb{H}^2$ *[5.11] is*

$$z \to w = D(z) = \frac{iz+1}{z+i}$$

To derive the metric (5.15) of the disc model, consider an infinitesimal vector dz emanating from z (in the half-plane) being amplitwisted to an infinitesimal vector dw = D' (z) dz emanating from w (in the disc). By definition, the hyperbolic length $d\hat{s}$ *of dw is the hyperbolic length of dz. Verify [5.15] by showing that*

$$\frac{2|dw|}{1-|w|^2} = \frac{|dz|}{Im\, z} = d\hat{s}.$$

$$[\qquad d\hat{s} = \frac{2}{1-r^2}\,\mathrm{ds} \qquad\qquad (5.15) \quad]$$

A derivation of [5.15] appears to be glossed over on the way to (44) in *VCA*, pp. 317-318, where it is shown on p. 318 that

$$\rho\, d\hat{s} = 1$$

and

$$2\,\rho\,\mathrm{ds} = (1-r)\times(1+r) = 1 - |\ z\ |^2$$

Vasco has posted derivations for these two equations on the VCA Forum.

But the notation must be adapted to our problem. Figures [36b] and [36d] in VCA use the symbol z for the arbitrary complex number on both the plane and the disc, whereas the current exercise represents it as w, confining z to the UHP. Using the w notation [36b] and [36d] and reasoning from VCA, we can write

$$\rho\, \mathrm{d}\hat{s} = 1 \Longrightarrow d\hat{s} = \frac{1}{\rho}$$

$$2\,\rho\,\mathrm{ds} = (1-r)\times(1+r) = 1 - |\ w\ |^2$$

Then

$$\rho = \frac{1-|w|^2}{2\,\mathrm{ds}}$$

$$d\hat{s} = \frac{1}{\rho} = \frac{2\,\text{ds}}{1-|w|^2}$$

$$d\hat{s} = \frac{2\,\text{ds}}{1-|w|^2} = \frac{2\,\text{ds}}{1-r^2} \qquad (5.15)$$

In the hyperbolic half-plane, we can write

$$d\hat{s} = \frac{\sqrt{\text{dx}^2+\text{dy}^2}}{y} = \frac{\text{ds}}{y} = \frac{|\text{dz}|}{\text{Im}\,z}$$

as shown in (5.7) and (5.8) on pp. 56 and 57. On p. 317, ¶ 3 we find

$$d\hat{s} = \frac{\text{ds}}{\text{Im}(z)}$$

so we conclude that $\frac{\text{ds}}{\text{Im}(z)}$ is the hyperbolic length of ds. Now we can write

$$d\hat{s} = \frac{2\,\text{ds}}{1-|w|^2} = \frac{2\,\text{ds}}{1-r^2}$$

where $r = |w|$, $(0 \le |w|, r \le 1)$.

Since the transformation is conformal, $d\hat{s}$ in the half plane in [36a] equals $d\hat{s}$ on the disc in [36b, d] (p. 318, VCA). From the exercise statement, *By definition, the hyperbolic length $d\hat{s}$ of dw is the hyperbolic length of dz.* So we can write

$$d\hat{s} = \frac{2\,\text{ds}}{1-|w|^2} = \frac{|\text{dz}|}{\text{Im}\,z}$$

And since this $d\hat{s}$ in the half-plane is the hyperbolic length of dz, which equals $d\hat{s}$ in the Beltrami-Poincare disc, we can infer that ds $= |\text{dw}|$ and write:

$$\frac{2|\text{dw}|}{1-|w|^2} = \frac{|\text{dz}|}{\text{Im}\,z} = d\hat{s}$$