Needham, VDGF, Chapter 7, Exercise 22, p. 89.
G. Palmer, July 1, 2022, 6:10 pm PDT

Out[681]=

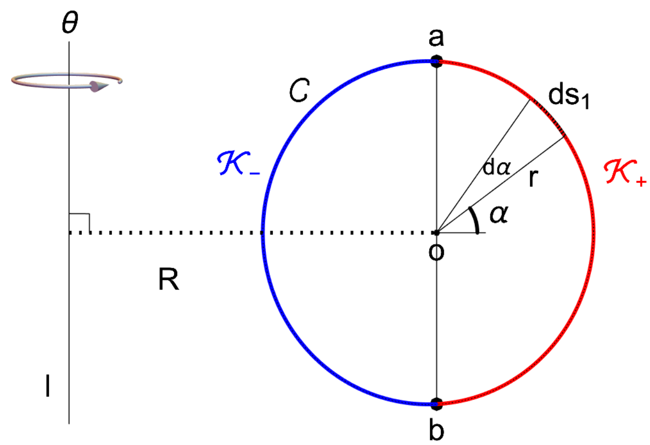

Figure 1. Based on figure in Exercise 23, p. 89

Out[682]=

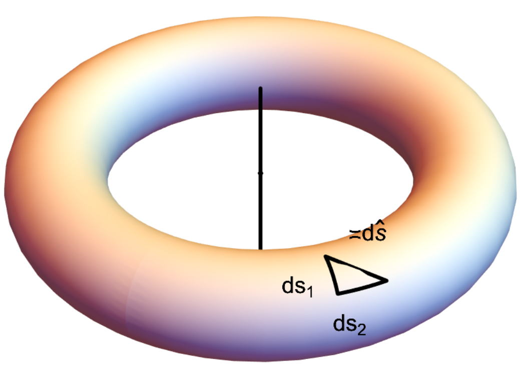

Figure 2. Torus from rotation of C about Z axis*

23. Curvature of the Torus. *Generate a torus T by rotating a circle C with centre o and radius r about a line l in the plane of C that is at distance R (>r) from o....*

(i) If α is the illustrated angle going round the circle, and θ is the illustrated angle of rotation about the axis of symmetry l, show geometrically that the metric is

$$d\hat{s}^2 = r^2\, d\alpha^2 + (R + r\cos\alpha)^2\, d\theta^2$$

Let ds_1 be an infinitesimal arc segment of a circle on the surface of the torus in a plane parallel to the vertical axis (Figures 1 and 2). Let ds_2 be an arc segment of a circle on the surface of the torus in a plane perpendicular to the vertical axis (Figure 2)*. Then this arc segment can also be described by $\mathcal{R}\theta$, where $\mathcal{R} = (R + r\cos\alpha)$. While ds_2 has height ($r\sin\alpha$), the length alone is sufficient to calculate $ds_2 = (R + r\cos\alpha)\, d\theta$ because ds_2 is the horizontal segment in this construction. Raising or lowering it does not alter the radial distance from the Z axis. Consequently

$$d\hat{s}^2 = ds_1^2 + ds_2^2 = r^2\, d\alpha^2 + (R + r\cos\alpha)^2\, d\theta^2$$

*In Figure 2, ds_2 is shown as a secant joining two vertices, but it should be plotted on the surface of the torus as a geodesic segment. The secant and the geodesic segment are ultimately equal as the triangle shrinks.

(ii) Use (4.10) to deduce that the curvature of the torus is

$$\mathcal{K} = \frac{1}{r(r+R\sec\alpha)}$$

Again note that the coordinates are orthogonal, so the metric has the form of (4.9)

$$d\hat{s}^2 = A^2\, du^2 + B^2\, dv^2$$

Let $A = r$, $du = d\alpha$, $B = (R + r\cos\alpha)$ and $dv = d\theta$. Then

$$\mathcal{K} = -\frac{1}{r(R+r\cos\alpha)}\left(\partial_\theta\left[\frac{\partial_\theta r}{R+r\cos\alpha}\right] + \partial_\alpha\left[\frac{\partial_\alpha(R+r\cos\alpha)}{r}\right]\right)$$

$$= -\frac{1}{r(R+r\cos\alpha)}\left(0 + \partial_\alpha\left[\frac{-r\sin\alpha}{r}\right]\right) = -\frac{1}{r(R+r\cos\alpha)}\,(0 - \cos\alpha)$$

$$= \frac{\cos\alpha}{r(R+r\cos\alpha)} = \frac{1}{r(R/\cos\alpha+r)} = \frac{1}{r(r+R\sec\alpha)}$$

Out[ ]=

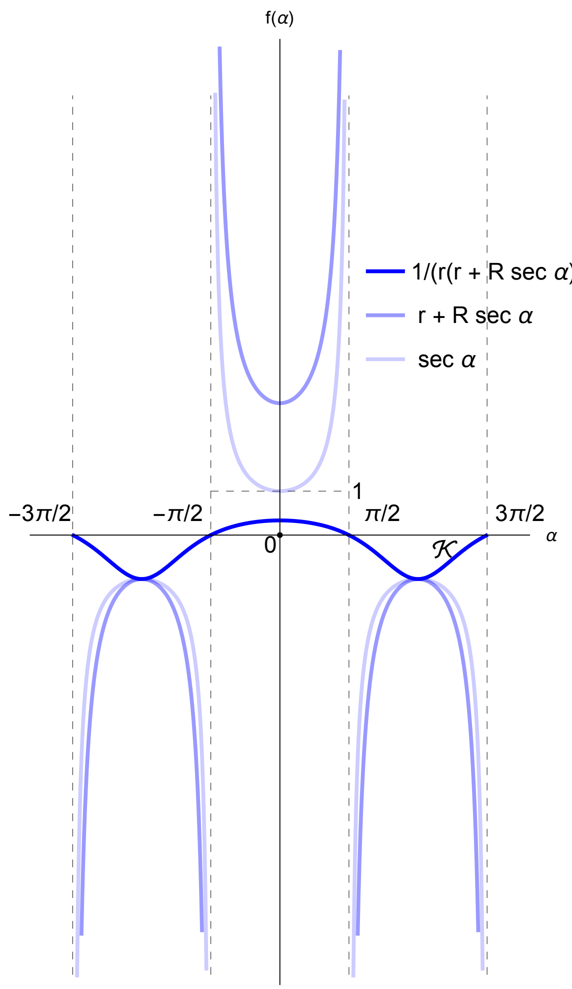

Figure 3. Secant and $\mathcal{K}$ on torus

(iii) Sketch the graph of sec α, then of (r + R sec α) and finally of $\mathcal{K}$ itself. Thereby confirm that this formula for $\mathcal{K}$ agrees with the empirical findings of Fig. [2.2] (p. 18).

One can see that the triangles on the near (outer) surface of the torus have excess positive curvature, while the triangles on the far (inner) surface of the torus have excess negative curvature. Since the triangles on the near outer surface lie at approximately $-\pi/2 \le \alpha \le \pi/2$, one can see by the dark blue line in Figure 3 that $\mathcal{K}$ is positive over that same range. Similarly, the triangles on the far inner surface lie in the range $\pi/2 \le \alpha \le 3\pi/2$ where the plot shows that $\mathcal{K}$ is negative. $\mathcal{K}$ is at a minimum at $\pm\pi \pm n2\pi$. It is at a maximum at $0 \pm n2\pi$.

(iv) Where on T would you expect to find $\mathcal{K} = 0$? Confirm this with the formula.

I would expect to find $\mathcal{K}$=0 at $\alpha = \pm\pi/2 \pm n2\pi$, where one encounters the boundary between the inside (hole side) of the torus, where $\mathcal{K}$ is negative, and the outside, where $\mathcal{K}$ is positive. It is at the boundary that sec α goes to infinity, causing $\mathcal{K}$ to go to 0. The inside and outside correspond respectively to the blue and red half-circles in Figure 1.

(v) We see from the formula that $\lim_{R\to\infty} \mathcal{K} = 0$*. Explain this geometrically.*

The interior axis of the torus is a circle with radius R. As $R \to \infty$, this circle becomes an infinite circle, which is a straight line. Hence, the curvature of the horizontal side ds_2 of the infinitesimal triangle shown in Figure 2 goes to zero according to $\mathcal{K} = 1/R^2$.

(vi) Use (4.12) to find the total area of T.

(4.12) is the area of a system with orthogonal coordinates (p. 38):

$$d\mathcal{A} = \text{AB dudv}$$

Use the same substitutions as in part (ii) to write

$$d\mathcal{A}_T = r \cdot (R + r\cos\alpha)\, d\alpha\, d\theta$$

$$\mathcal{A}_T = \int\int [r \cdot (R + r\cos\alpha)]\, d\alpha\, d\theta$$

where the ranges of $d\alpha$ and $d\theta$ are both 0 to 2 π. Let the constant of integration be 0.

$$= \int\int r \cdot R\, d\alpha\, d\theta + \int\int r^2 \cos\alpha\, d\alpha\, d\theta + 0.$$

$$= r\,R\,2\,\pi \int d\theta + r^2 \int \sin\alpha\, d\theta$$

$$= r\,R(2\,\pi)^2 + 0 = 4\,\pi^2\,r\,R$$

(vii) Check that your formula for the area agrees with the prediction of ***Pappus's Centroid Theorem****....*

Pappus's Centroid Theorem: *the surface area A of a surface of revolution generated by rotating a plane curve C about an axis external to C and on the same plane is equal to the product of the arc length s of C and the distance d traveled by the geometric centroid of C*.

The arc length s of C is the circumference of the cross section of the torus tube: $s = 2\,\pi\,r$. The distance traveled by the geometric centroid of C is $2\,\pi\,R$.

Then $\mathcal{A}_T = (2\,\pi\,r) \times (2\,\pi\,R) = 4\,\pi^2\,r\,R$, as in part (vi).

(viii) Show that the total curvature of the torus vanishes:

$$\oint\!\!\oint_T \mathcal{K}\, d\mathcal{A} = 0$$

$$\mathcal{K} = \frac{1}{r(r+R\sec\alpha)}$$

$$\mathcal{K}_{\text{Total}} = \int\int \frac{1}{r(r+R\sec\alpha)}\, d\alpha\, d\theta$$

Begin with the interior integral. If the result equals 0, nothing further will be needed.

$$\mathcal{K}_{\mathrm{d}\alpha} = \int \frac{1}{r(r+R\sec\alpha)}\, d\alpha = \int_{-\pi/2}^{\pi/2} \frac{1}{r(r+R\sec\alpha)}\, d\alpha + \int_{\pi/2}^{3\pi/2} \frac{1}{r(r+R\sec\alpha)}\, d\alpha$$

The two integrals represent the curvature on two segments of the small circumference of the torus. They correspond respectively to the red and blue half circles in Figure 1. These continuous smooth segments are equal in length and in magnitude of angles (α) spanned, and they are symmetrical in shape about a vertical axis ab (Figure 1) intersecting the circular central axis within the torus. And since their curvature is described by the same formula, they have equal curvature except for sign of $\mathcal{K}_{\mathrm{d}\alpha}$. But we know from Figure [2.2] that the inner surface has negative curvature and the exterior surface has positive curvature. Therefore, the two terms of the integral sum to zero, so $\mathcal{K}_{\mathrm{d}\alpha} = 0$ and $\oint\!\!\oint_T \mathcal{K}\, d\mathcal{A} = 0$.