Needham, VDGF, Chapter 7, Exercise 22, p. 89.

G. Palmer, July 6, 2022, 11 am PDT

***22. Surfaces of Revolution of Constant Curvature***. *Imagine a particle traveling along a curve in the (x, y)-plane at unit speed, and let its position at time t be [x(t), y(t)]. Now imagine rotating this plane through the angle θ about the x-axis. As θ varies from 0 to 2π, the curve sweeps out a surface of revolution.*

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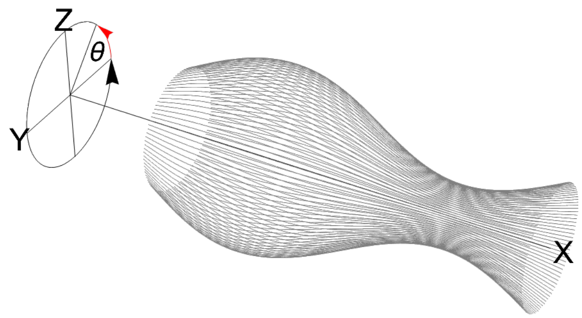


`Figure 1. Surface of Revolution of curve (t,.5 sin(t)+1)`

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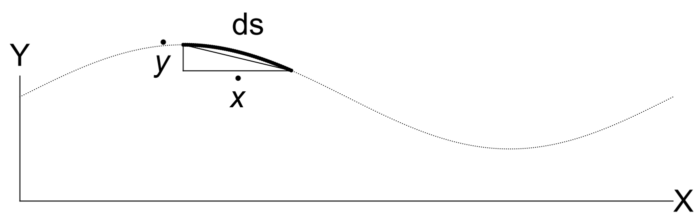


`Figure 2. Generator curve for surface of revolution`

*(i) Explain why* $\dot{x}^2 + \dot{y}^2 = 1$*, where the dot represents the time derivative.*

See Figure 2. Another way of writing this equation is $(dx/dt)^2 + (dy/dt)^2 = 1$. Since $(ds/dt)^2 = (dx/dt)^2 + (dy/dt)^2$, where ds is distance traveled on the curve in the (x,y)-plane, this is another way of showing that the particle has unit speed. This does not necessarily mean that a particle traveling in a random direction on the surface of revolution will have unit speed. Note that ds = *v t*. Vasco (P. c. VDGF Forum) points out that one can write

$$dx^2 + dy^2 \asymp ds^2 = 1^2 \cdot dt^2$$

since the particle speed is 1. Then it follows that

$$(dx/dt)^2 + (dy/dt)^2 = \dot{x}^2 + \dot{y}^2 = 1^2 = 1$$

This has implications for parts (ii) and (v).

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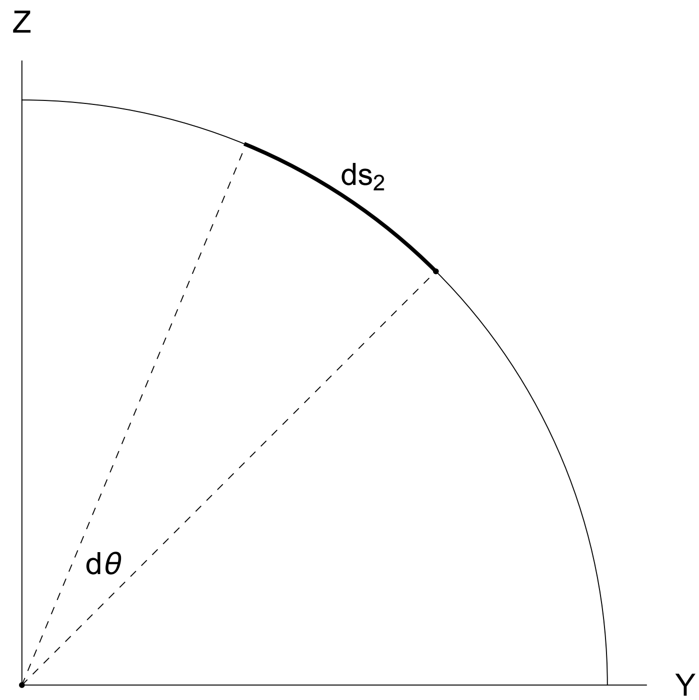


Figure 3. $ds_2$

*(ii) Show geometrically that the metric of the surface is $d\hat{s}^2 = dt^2 + y^2\, d\theta^2$.*

The viewpoint of Figure 3 is somewhere on the positive x axis. The plot shows a quarter segment of a circle in the plane {Z, y(t)}. The circle is a cross-section of the surface of revolution taken at x(t). One can see on the plot that $ds_2 = y(t)\, d\theta$. The orthogonal coordinate needed for the surface metric is $ds_1^2 \asymp ds^2 = dx^2 + dy^2 = dt^2$ (from figure 2 and part (i)). So one can write

$$d\hat{s}^2 \asymp ds_1^2 + ds_2^2 = dt^2 + y^2\, d\theta^2$$

and change the asymptote symbol to an equal sign.

*(iii) Deduce from that (4.10) that $\mathcal{K} = -\ddot{y}/y$. Let A = 1, B = y, u = t, and v = θ.*

$$\mathcal{K} = -\frac{1}{\mathrm{AB}}\left(\partial_v\left[\frac{\partial_v A}{B}\right] + \partial_u\left[\frac{\partial_u B}{A}\right]\right)$$

$$\mathcal{K} = -\frac{1}{y}\left(\partial_\theta\left[\frac{\partial_\theta 1}{y(t)}\right] + \partial_t\left[\frac{\partial_t y(t)}{1}\right]\right) = -\frac{1}{y}\left(0 + \ddot{y}\right) = -\ddot{y}/y$$

*(iv) Find the general solution y(t) in each of the three cases of constant curvature $\mathcal{K} = 0$, $\mathcal{K} = 1$, and $\mathcal{K} = -1$.*

$\mathcal{K}$ **= 0**: $-\ddot{y}/y = 0 \Longrightarrow \ddot{y} = 0 \Longrightarrow \dot{y}$ = const., so y is a straight line mx + b or it is the degenerate case y = 0.

$\mathcal{K}$ **= *1***: $-\ddot{y}/y = 1 \Longrightarrow y = -\ddot{y}$ or $y + \ddot{y} = 0$

For equations of this type, $y(t) = A \sin t + B \cos t$ is a general solution.

$\mathcal{K}$ **= −1 :** $-\ddot{y}/y = -1 \Longrightarrow y - \ddot{y} = 0$

$e^{rt} - r^2 e^{rt} = 0 \Longrightarrow 1 - r^2 = 0 \Longrightarrow r^2 - 1 = 0 \Longrightarrow r = \pm 1$

Then $y(t) = \text{A}e^t + \text{B}e^{-t}$.

*(v) With the assistance of part (i), calculate the velocity of the particle in each of the three cases.*

The exercise does not stipulate that the particle changes direction, so the particle should continue to traverse a curve in the {x, y}-plane and there should be no change in speed. The velocity is a vector of length 1 that points in the direction of the tangent at each point p on the curve. Using part (i), the vector can be written $v(t) = (\dot{x}(t), \dot{y}(t))$ with slope $m(t) = \dot{y}(t)/\dot{x}(t)$. Then for the cases of $\mathcal{K}$,

$\mathcal{K}$ **= 0**

$\dot{y}(t) = m$ (which is a constant)

$v(t) = (\dot{x}(t), m)$

$\dot{x}^2 + m^2 = 1 \Longrightarrow \dot{x}(\text{t}) = \sqrt{1 - m^2} \Longrightarrow v(t) = \left(\sqrt{1 - m^2}, m\right)$

$\mathcal{K}$ **= 1**

$y(t) = A \sin t + B \cos t$

$\dot{y}(t) = A \cos t - B \sin t$

$\text{v(t)} = (\dot{x}(t), A \cos t + B \sin t)$

$= \left(\pm \sqrt{1 - (A \cos t - B \sin t)^2}, \text{A cos t -B sin t}\right)$

**$\mathcal{K}$ = –1**

$y(t) = \mathrm{A}e^{t} + \mathrm{B}e^{-t}$

$\dot{y}(t) = \mathrm{A}e^{t} - \mathrm{B}e^{-t}$

$\mathrm{v(t)} = (\dot{x}(t),\ \mathrm{A}e^{t} - \mathrm{B}e^{-t})$

$= (\pm\sqrt{1 - (\mathrm{A}e^{t} - \mathrm{B}e^{-t})^{2}},\ \mathrm{A}e^{t} - \mathrm{B}e^{-t})$

*(vi) Sketch some solution curves....*

In Exercise 16 we sketched the pseudosphere, which in part (i) was shown to have $\mathcal{K}$ = –1 (iii). In Figure 1, above, a surface with both positive and negative curvature is sketched. To obtain a surface with $\mathcal{K}$ = 1, it would be easy enough to limit a surface based on a standard cosine curve to $-\pi/4 \leq x \leq \pi/4$. Where $\mathcal{K}$ = 0, it follows that the curve in the {x,y}-plane is a straight line (from part (iv)), the surface of rotation should be a cylinder or a conical section as in Figure 4. However, these have curvature given by the rotation of the generator. In Figures labeled 4, 5, and 6, generators are shown as thick, red lines.

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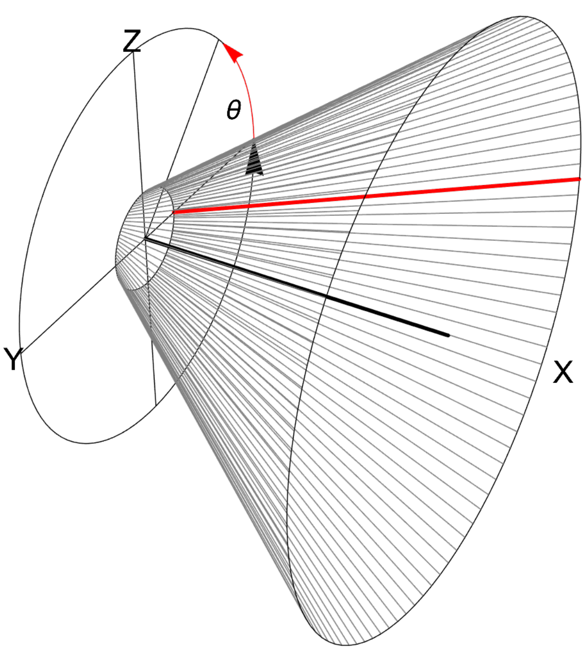


Figure 4. Surface of Revolution of line $\mathcal{K}$=0, m=0.75

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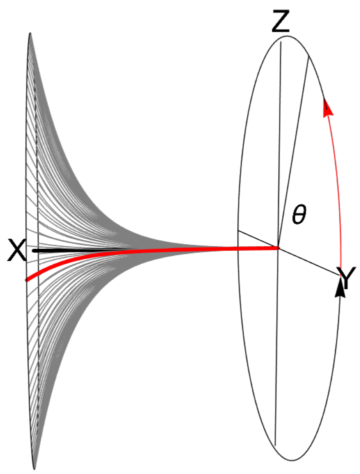


Figure 5a. Surface of Revolution
$\mathcal{K}$=−1, (t, $e^t$/100)

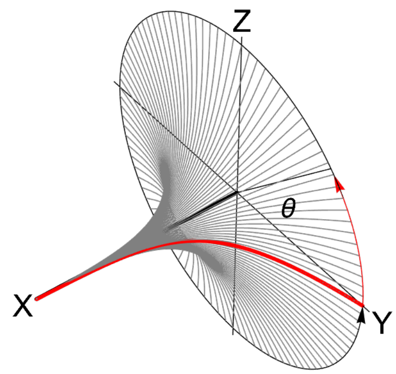


Figure 5b. Surface of Revolution
$\mathcal{K}$=−1, (t, $5e^{-t}$)

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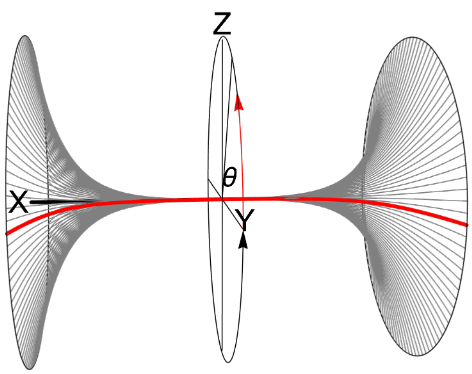


Figure 5c. Surface of Revolution
$\mathcal{K}$=−1, (t, $e^t$/100) and (−t, $5e^t$)

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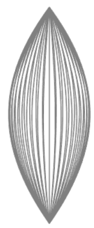

Figure 6a. Surface of Revolution
$\mathcal{K}$ = 1, {t, 2 cos t − 2 cos $\pi$/4}
−$\pi$/4 ≤ t ≤ $\pi$/4

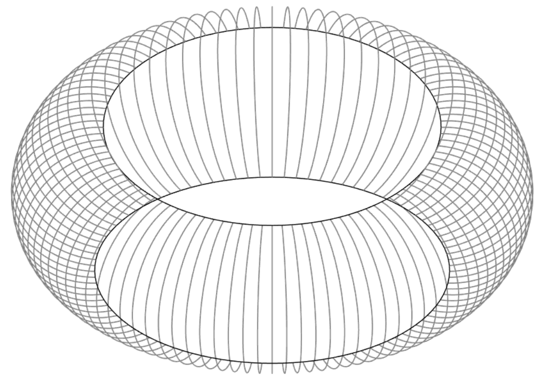

Figure 6b. Surface of Revolution
$\mathcal{K}$ = 1, {cos t, sin t + 2}
0 ≤ t ≤ $\pi$