Needham, VDGF, Chapter 7, Exercise 20, p. 29.
G. Palmer, June 21, 2022, 12:30 am PDT

20. Geodesics of the Pseudosphere. *Using (1.7)*

(i) Use the Beltrami-Poincaré upper half-plane to verify mathematically that the tractrix generators are the only geodesics that extend infinitely upward.

Figure [5.5] shows a tractrix generator and a geodesic on the hyperbolic plane. The tractrix generator is vertical and extends infinitely upward on the y-axis. A typical geodesic is a half-circle on the hyperbolic plane with a maximum point were f' = 0. This is the point where ds = 0, so $d\hat{s} = 0/y = 0$ at this point. Since the map is conformal, the maximum point on the map (zero slope) must correspond to the maximum point on the pseudosphere, though they are not necessarily at the same height. Confined to the tractrix and its image in $\mathbb{H}^2$, $d\hat{s}$ and dy are never 0, so there is nothing preventing the tractrix generator from extending infinitely.

Out[295]=

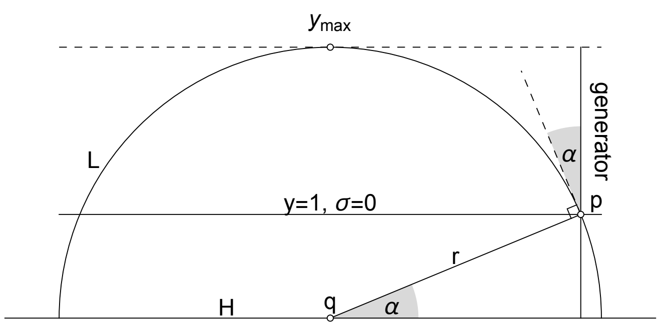

Figure 1. σ_{max} for α, Poincaré UHP

(ii) Let L be a typical geodesic, and let α be the angle between L and the tractrix generator at the point where L hits the rim at $\sigma = 0$. Show that the maximum distance σ_{max} that L travels up the pseudosphere is given by $\sigma_{max} = |\ln \sin \alpha|$.

By using complementary angles one can deduce that the shaded angle at q equals the shaded angle at p. Then, since the rim is at y = 1, it follows that $\sin \alpha = 1/r$, and $r = 1/\sin \alpha$. Figure 1 shows that $y_{max} = r$.

$y_{max} = e^{\sigma_{max}/R} = e^{\sigma_{max}}$, because R = 1.*

$\Longrightarrow e^{\sigma_{max}} = 1/\sin \alpha$

$\sigma_{max} = -\ln \sin \alpha$

But we are only interested in σ_{max} as a positive length, so

$$\sigma_{max} = | \ln \sin \alpha |$$

Note, if some other value of R were used, the equation would become:

$$\sigma_{max} = R \, | \ln \sin \alpha |$$

**vide* box p. 56, top. In the text VDGF used here, R first appears on p. 52 as the length of the object being dragged to form the tractrix, so R is a constant and belongs to the tractrix. R is not the same as the r shown on $\mathbb{H}^2$ in Figure 1.

There seems no point to plotting the distance σ_{max} on $\mathbb{H}^2$, because the conformal map does not preserve distances, only angles. This same exercise appears in *Visual Complex Analysis*, Chapter 6, Exercise 16.