Needham, VDGAF, Chapter 7, Exercise 2, p. 83
Palmer, February 4, 2022, 10 pm PST

2. Central Projection. Consider the central projection of the sphere shown in [4.1].

(i) Prove that an infinitesimal circle on the sphere is distorted into an ellipse whose shape is given by (4.3).

Let ϕ be the angle from c (center at equator, as in [4.2]) to any point on the great circle GC that passes through c and any point of latitude, such as the center of the small circle SmC (as in [4.1]) on the hemisphere $\mathcal{S}$. Let $\delta\phi$ be the angle between a ray from c to the center of SmC and a ray from c to one of the two points of intersection of SmC and GC. Since $r_C = R\delta\phi$, as $\delta\phi$ shrinks to 0, so does r_C.

Let $r_{\min}$ and r_{maj} be the minor and major axes of the ellipse $\in$. Then, as $\delta\phi$ shrinks to 0, the vertical axis of SmC approaches the meridian segment $R\delta\phi$. This axis projected onto the ellipse $\in$ *becomes* $r_{\text{maj}} = \delta r$, which from p. 33 and [4.2], we know to be described by the equation

$$\frac{\varepsilon}{\delta r} \asymp \frac{R}{H}$$

From [4.2], one can see that in this case that $\varepsilon \asymp H\,\delta\phi$. Then

$$r_{\text{maj}} = \delta r \asymp \frac{\varepsilon H}{R} = \frac{(H\,\delta\phi)\,H}{R} = \frac{H^2\,\delta\phi}{R}$$

On the horizontal (latitudinal) axis of SmC, the radius remains constant, so it again approaches $R\delta\phi$, but on the plane, the distance becomes $r_{\min} = H\delta\phi$.

$$\frac{r_{\text{maj}}}{r_{\min}} \asymp \frac{\left(\frac{H^2\,\delta\phi}{R}\right)}{H\,\delta\phi} = \frac{H}{R} = \sec(\phi)$$

which is a version of (4.3). The hemisphere is differentiable over its surface, so the small circle on the surface is differentiable over its principal theta domain. The projection of the circle onto the plane must also be differentiable with dimensions on rays versus lines parallel to the tangent to the sphere in the ratio H/R. This proves that the infinitesimal circle is distorted into an ellipse by projection from the centre of the sphere, and that this ellipse is described by (4.3).

(ii) Explain why this distortion is symmetrical: if we instead start with an infinitesimal circle in the plane, the image of the sphere is an ellipse of the same shape.

The term “symmetrical” strikes me as ambiguous between (1) the idea that projection of the circle from the plane to the sphere also produces an ellipse, and (2) the idea that the ellipse is symmetrical about both axes under this construction. I will consider (2) first and then use the answer to explain (1).

Consider a great circle centred on c and passing through the centre q of SmC at an angle ϕ so that the GC is tangent to a line of latitude L intersecting q. This great circle projects onto the plane as a straight line parallel to the tangent of L at q. As dϕ goes to zero, so does dθ and the radius of SmC. Since the radius of SmC is symmetrical about the centre of SmC on this latitudinal orientation, the radius $r_{\min}$ of the ellipse must also be symmetrical about a ray from the centre of the ellipse to the Z axis of the sphere.

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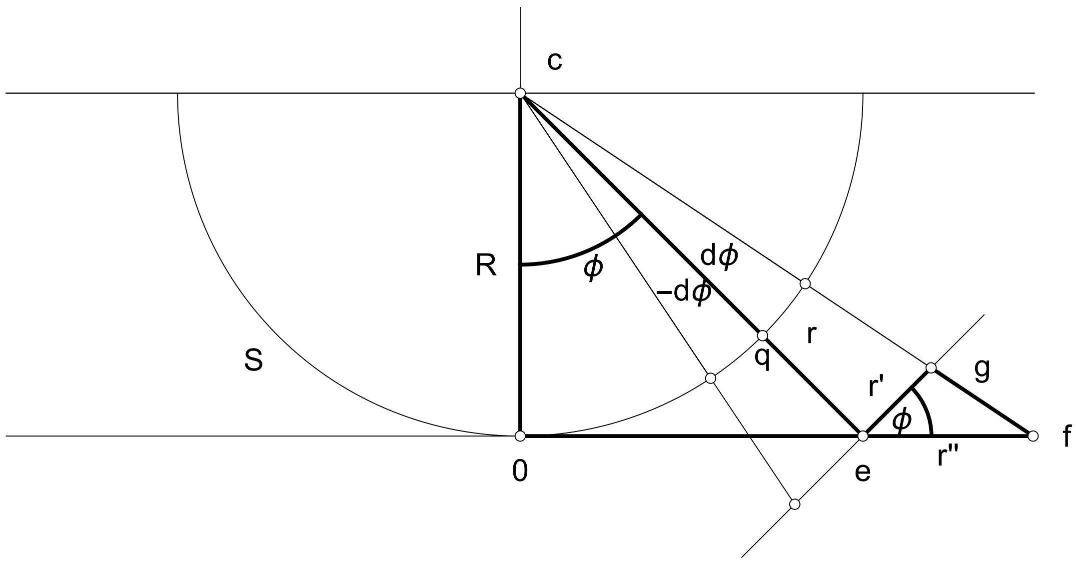


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Figure 1. Central projection of circle
from sphere to plane tangent to S at 0
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The framework for the following and Figure 1 was adapted from “Mapping the sphere” (https://math-.rice.edu/~polking/cartography/cart.pdf). See the section on “Distortion in Maps”.

Infinitesimal regions of S approximate planes and infinitesmal segments great circles on the sphere approximate straight lines. This perspective seems to make projection to or from a circle on the sphere equivalent to projection from a circle drawn on a plane orthogonal to a ray from a central point. From Figure 1, one can see that, where $r \asymp \mathrm{R}d\phi$,

$$\frac{r'}{r} \asymp \frac{|e-c|}{R} = \frac{1}{\cos\phi} \Longrightarrow r' \asymp \frac{r}{|\cos\phi|} \qquad (1)$$

Note that on the horizontal (latitudinal) axis, dθ = dϕ, so this is all that is needed for calculation of the minor axis: $r_{\min} = r' \asymp \frac{r}{|\cos\phi|}$.

Finding the major (longitudinal) axis requires one more step. Triangles 0ce and efg are similar. Since dϕ, r, and r' are infinitesimal, r and r' approximate chords, and the angles at q, the angle qeg, and the angle egf of the triangle efg approximate right angles. One can write

$$\cos\phi \asymp \frac{r'}{r''} \Longrightarrow r'' \asymp \frac{r'}{|\cos\phi|}$$

Now subsitute (1) for r' :

$$r'' \asymp \frac{\left(\frac{r}{\cos\phi}\right)}{\cos\phi} = \frac{r}{\left|\cos^2\phi\right|} \qquad (2)$$

Thus, the image of SmC is an ellipse with semi-axes $r_{\text{min}} = r / \mid \cos\phi \mid$ and $r_{\text{maj}} = r \big/ \cos^2\phi$.

To explain why the projection of the circle from the plane to the sphere is an ellipse, one can take the inverses of equations (1) and (2).

$$r_{\text{min}} = r' \mid \cos\phi \mid$$

$$r_{\text{maj}} = r'' \cos^2\phi$$

This shows that that r_{min} varies with $\mid \cos\phi \mid$ and that r_{maj} shrinks with $\cos^2\phi$ as ϕ goes to $\pm\pi/2$. Hence, the result is an ellipse.