Needham, VDGF, Chapter 7, Exercise 17, p. 88.

G. Palmer, June 17, 2022, 1:40 pm PDT

17. ***Cylindrical-Polar Area on Sphere.*** *If we employ the cylindrical-polar coordinates of the previous exercise, the element of area on the northern hemisphere of the unit sphere is,*

$$d\mathcal{A} = \frac{r\,dr\,d\theta}{\sqrt{1-r^2}}$$

Out[∘]=

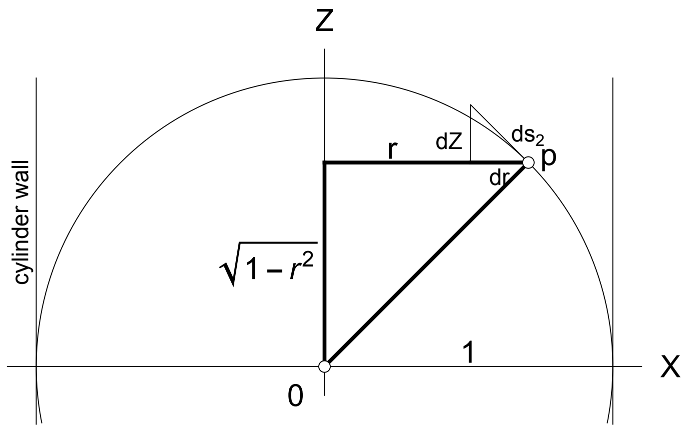


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Figure 1. Cylindrical coordinates
on cross-section of sphere at θ = 0
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(i) Prove this by calculation.

Read r as distance from the Z axis (Figure 1). “p” is an arbitrary point on the hemisphere at $(r,\ 0)$ in cylindrical coordinates $(r,\ \theta)$. It is obvious that the horizontal component of area on the cylinder can be written

$$ds_1 = r\,d\theta,$$

which is not shown here, but see Figure 2b and p. 33, paragraph 2. Then one can obtain the orthogonal coordinate ds_2 as follows:

$$ds_2^2 = dZ^2 + dr^2$$

$$dZ = \partial_r(Z)\,dr = \partial_r\left(\sqrt{1-r^2}\right)dr = -\frac{r\,dr}{\sqrt{1-r^2}}$$

$$ds_2^2 = \left(-\frac{r\,dr}{\sqrt{1-r^2}}\right)^2 + dr^2 = \frac{r^2\,dr^2}{1-r^2} + \frac{(1-r^2)\,dr^2}{1-r^2} = \frac{dr^2}{1-r^2}$$

Consequently, $ds_2 = \frac{dr}{\sqrt{1-r^2}}$.

$$d\mathcal{A} = ds_1\, ds_2 = (r\, d\theta)\, \frac{dr}{\sqrt{1-r^2}} = \frac{r\, dr\, d\theta}{\sqrt{1-r^2}}$$

This is what we wanted.

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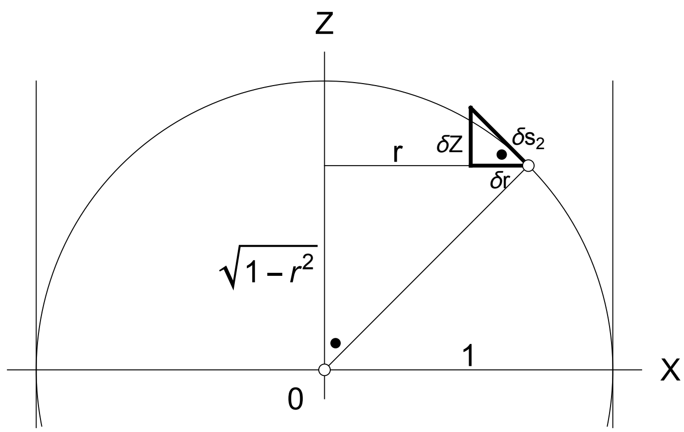

Figure 2a. Very small triangle
on cross-section of sphere at θ = 0

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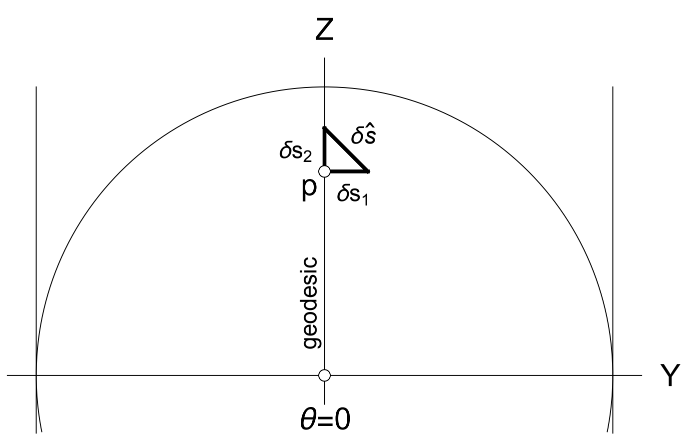

Figure 2b. Very small triangle at θ = 0
on surface of sphere

(ii) Prove this geometrically, using Newtonian ultimate equalities.

In figure 2a, the hypotenuse of the small bold triangle is ultimately equal to to the spherical segment $\delta\hat{s}$ to which it corresponds as the triangle is allowed to shrink to infinitesimal size. So one can write

$$\frac{\delta Z}{\delta r} \asymp \frac{r}{\sqrt{1-r^2}} \Longrightarrow \delta Z \asymp \frac{r\delta r}{\sqrt{1-r^2}}$$

$$\delta s_2^2 \asymp \delta Z^2 + \delta r^2 \qquad \text{(Figure 2a)}$$

$$\delta s_2 \asymp \sqrt{\delta Z^2 + \delta r^2} = \sqrt{\frac{r^2\,\delta r^2}{1-r^2} + \delta r^2}$$

$$\asymp \sqrt{\frac{r^2\,\delta r^2 + \delta r^2\left(1-r^2\right)}{1-r^2}} = \frac{\delta r}{\sqrt{1-r^2}}$$

In figure 2b, the horizontal spherical segment is ultimately similar to the base of the small bold triangle as the sides are permitted to shrink, so one can apply $\delta s_1 \asymp r d\theta$. Change the very small δ's to infinitesimal d's to confirm the expected result.

$$d\mathcal{A} = ds_1\, ds_2 = \frac{r\,dr\,d\theta}{\sqrt{1-r^2}}$$

(iii) Integrate this to obtain the area of the complete hemisphere.

$$d\mathcal{A} = \int_0^1 \int_0^{2\,\text{Pi}} \frac{r\,dr\,d\theta}{\sqrt{1-r^2}}\, d\theta\, dr = \int_0^1 \frac{r}{\sqrt{1-r^2}} \int_0^{2\,\text{Pi}} d\theta\, dr$$

$$= \int_0^1 \frac{r}{\sqrt{1-r^2}}\, 2\,\pi\, dr = 2\,\pi\left[-\sqrt{1-r^2}\,\right]_0^1 = 2\,\pi(0-(-1)) = 2\,\pi$$

As expected, this is half the area of the unit sphere, $4\,\pi r^2 = 4\,\pi$.