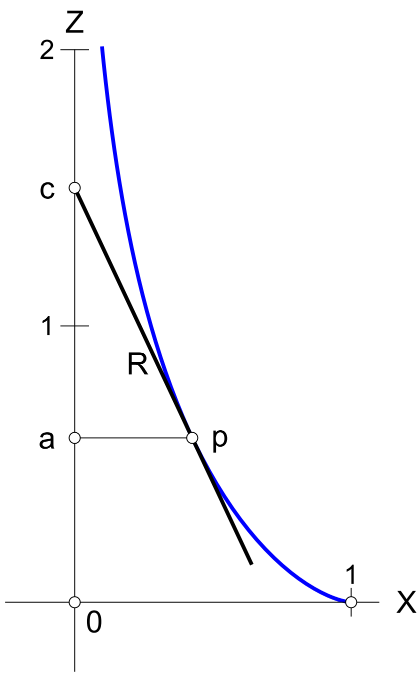
Needham, VDGF, Chapter 7, Exercise 16, p. 88.
G. Palmer, June 13, 2022, 2:30 PDT

Out[653]=



`Figure 1. R = const. over tractrix`

16. Tractrix Parameterization. In (X,Y,Z)-space let us switch to cylindrical polar coordinates, i.e. ordinary (r, θ) *polar coordinates in the (x,y)-plane, supplemented by Z. The Euclidean metric in this space is therefore*

$$ds^2 = dr^2 + r^2\, d\theta^2 + dZ^2.$$

In the plane $\theta = 0$*, i.e., the (X,Z)-plane, consider the curve traced by a particle whose position at time t is*

$$X = r = \operatorname{sech} t, \qquad Z = t - \tanh t.$$

(i) Show that this curve is the tractrix: the distance along the tangent to the Z-axis is fixed. What is this fixed distance?....

X on the pseudosphere in [5.2] corresponds to X on the curve of the (X, Z) plane. Y on the pseudosphere corresponds to Z on the (X, Z) plane.

Now we wish to find the tangent of this curve and find the Z intercept at X = 0. Then, Pythagoras will help with the rest.

tractrix = (X, Z) = (sech t, t – tanh t)

$$\left(\frac{dX}{dt}, \frac{dZ}{dt}\right) = \left(-\operatorname{sech} t \tanh t,\ 1 - \operatorname{sech}^2 t\right) = \left(-\operatorname{sech} t \tanh t,\ \tanh^2 t\right)$$

Use the equation for a line mx + c.

$$m(t) = \frac{dZ(t)}{dX(t)} = \frac{\tanh^2 t}{-\operatorname{sech} t \tanh t} = \frac{\tanh t}{-\operatorname{sech} t} = -\tanh t \cosh t = -\sinh t$$

Looking at the right triangle with vertices ***a*** on Z-axis at height of **p**, ***c*** (Z-intercept of tangent), and **p**, an arbitrary point on the curve at time t, one can write an equation for the tangent line:

$$Z = m(t)\, X(t) + c$$

Then

$$-(c - a) = m(t)\, X(t)$$

$$R^2 = (-(c-a))^2 + X(t)^2 = (m(t)\, X(t))^2 + X(t)^2$$

$$= (-\sinh t)^2 \operatorname{sech}^2 t + \operatorname{sech}^2 t$$

$$= \operatorname{sech}^2 t \left(\sinh^2 t + 1\right) = \frac{1}{\cosh^2 t} \cosh^2 t = 1$$

Hence, R = 1, and we have shown that the curve (sech t, t – tanh t) is a tractrix with constant distance 1 along the tangent at any point p to the Z-axis. But if we know this is a tractrix, we know the distance is 1, because the initial distance is the distance from sech(0) to the origin. sech(0) = 1.

(ii) Deduce that if we rotate the curve about the Z-axis, we obtain the pseudosphere of unit radius. show that the surface's metric is

$$d\hat{s}^2 = \tanh^2 t\, dt^2 + \operatorname{sech}^2 t\, d\theta^2$$

The Euclidean metric is given as $ds^2 = dr^2 + r^2\, d\theta^2 + dZ^2$.

$r = X = \operatorname{sech} t$

$dr = \frac{dX}{dt} = -\operatorname{sech} t \tanh t$ (from part (i))

$$\frac{dZ}{dt} = \tanh^2 t \qquad \text{(from part (i))}$$

Substitute r, dr, and $\frac{dZ}{dt}$ into the Euclidean equation

$$\begin{aligned} d\hat{s}^2 &= (-\text{sech}\, t \tanh t)^2\, dt^2 + sech^2\, t\, d\theta^2 + \left(\tanh^2 t\right)^2 dt^2 \\ &= \left(\text{sech}^2\, t \tanh^2 t + \left(\tanh^2 t\right)^2\right) dt^2 + sech^2\, t\, d\theta^2 \\ &= \tanh^2 t \left(\text{sech}^2\, t + \tanh^2 t\right) dt^2 + sech^2\, t\, d\theta^2 \\ &= \tanh^2 t\, dt^2 + sech^2\, t\, d\theta^2 \end{aligned}$$

This is what we wanted. This equation can be just as easily derived by plotting two small triangles, the first on the tractrix and its plane where $d\sigma^2 \asymp dZ^2 + dX^2$ and the second on the pseudosphere where $d\hat{s}^2 \asymp d\sigma^2 + X^2\, d\theta^2$. Substituting for $d\sigma$ and removing the $\asymp$ symbol, the equation becomes $d\hat{s}^2 = dZ^2 + dX^2 + X^2\, d\theta^2$, which is equivalent to the Euclidean equation.

(iii) Use (4.10) to confirm that $\mathcal{K} = -1$.

Taking the result of part (ii) as an instance of (4.9), let A = tanh t and let B = sech t. Let u = t and v = θ. Then

$$\mathcal{K} = -\frac{1}{\tanh t\, \text{sech}\, t}\left(\partial_\theta \frac{\partial_\theta \tanh t}{\text{sech}\, t} + \partial_t \frac{\partial_t \text{sech}\, t}{\tanh t}\right) = -\frac{1}{\tanh t\, \text{sech}\, t}\left(\partial_\theta \frac{0}{\text{sech}\, t} + \partial_t \frac{-\text{sech}\, t \tanh t}{\tanh t}\right)$$

$$= -\frac{1}{\tanh t\, \text{sech}\, t}\left(0 + \partial_t(-\text{sech}\, t)\right) = -\frac{\tanh t\, \text{sech}\, t}{\tanh t\, \text{sech}\, t} = -1$$

(iv) Show that if we now introduce new coordinates $x = \theta$ and $y = \cosh t$, then the metric assumes the standard for (5.7) [Which is $d\hat{s} = ds/y$].

$$d\hat{s}^2 = tanh^2\, t\, dt^2 + sech^2\, t\, d\theta^{\,2} = \frac{sinh^2\, t\, dt^2}{cosh^2\, t} + \frac{dx^{\,2}}{cosh^2\, t}$$

$$= \frac{(d(\cosh t)/dt)^2\, dt^2 + dx^{\,2}}{y^2} = \frac{dy^2 + dx^{\,2}}{y^2}$$

$$d\hat{s} = \frac{\sqrt{dy^2 + dx^{\,2}}}{y} = \frac{ds}{y}$$

This is what we wanted.