Needham, VDGF, Chapter 7, Exercise 15, p. 88.
G. Palmer, June 6, 2022, 10:50 am PDT

Out[ ]=

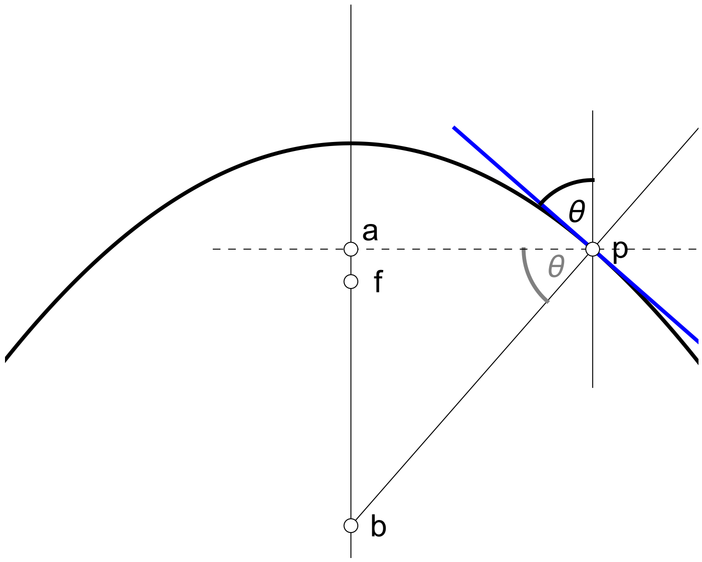

Figure 1. Generalized parabola $y = -(1/(4|f|))x^2$
The blue line is tangent to the parabola. Line bp is normal to the parabola at p.

15. If the velocity of light in the (x,y) plane is $v = 1\big/\sqrt{1-y}$*, show that the geodesics are parabolas. Describe these parabolas.*

This exercise appears to pertain to *Section 5.5 Using Optics to Find the Geodesics*. The (x,y) plane must then refer to the hyperbolic plane $\mathbb{H}^2$ in [5.5] with metric $d\hat{s} = \frac{ds}{y}$. On p. 60 it is explained that one can substitute for v(y) in equation 5.12.

$$\frac{\sin\theta(y)}{v(y)} = k \qquad (5.12)$$

Where light takes the shortest path on $\mathbb{H}^2$, θ is an angle made by the tangent of the curve with a vertical line at any point y on the hyperbolic plane. See [5.7] and Figure 1. In this construction, the parabola corresponds to the path of incident light.

Since it is given that $v = 1\big/\sqrt{1-y}$, geodesics in the Beltrami-Poincaré half plane model satisfy (1).

$$k = \frac{\sin\theta(y)}{1\big/\sqrt{1-y}} = \sqrt{1-y}\,\sin\theta(y) \qquad (1)$$

Since k must be a real number, y ≤ 1, which shows that the construction does not map to the pseudosphere. Rewrite (1)

$$\sin\theta(y) = k \Big/ \sqrt{1-y}$$

and let the hypotenuse of the triangle abp be $\sqrt{1-y}$. The normal line bp creates a new θ angle for which the opposite side has length $k \Big/ \sqrt{1-y}$. Then it must also be the case that

$$\cos\theta(y) = \frac{x}{\sqrt{1-y}}$$

where $x = \mid p - a \mid$ in Figure 1. Then using $c^2 + s^2 = 1$,

$$\frac{k^2}{1-y} + \frac{x^{2.}}{1-y} = 1$$

$$k^2 + x^2 = 1 - y$$

$$y = 1 - k^2 - x^2$$

Vasco noted (P. C., VDGF, Forum) that this can be written $y + \left(k^2 - 1\right) = -x^2$, “which is the equation of a parabola with focus at $\left(0,\ k^2 - 1 - 1/4\right)$.” With this focus, it must be that a characteristic of the family is that $k^2 \leq 2.25$, so $\left| k \right| \leq \sqrt{2.25}$. But in fact, both k and x must be constrained to ensure that $0 \leq y \leq 1$, because y cannot dip below the horizon of the hyperbolic plane at y = 0. Each of the steps below implies the following constraint.

$$x^2 \leq \left(1 - k^2\right) \implies \left| x \right| \leq \sqrt{1-k^2} \implies \left| k \right| < 1 \implies \left| x \right| \leq 1$$

The practical result is the white bounding box of Figure 2. The plotting of the parabolas reflects the above constraints on both x and k except that k is here limited to positive values less than 1 to make the relation of k to the parabolas more obvious. No values of $y \leq 0$ occur.

Out[ ]=

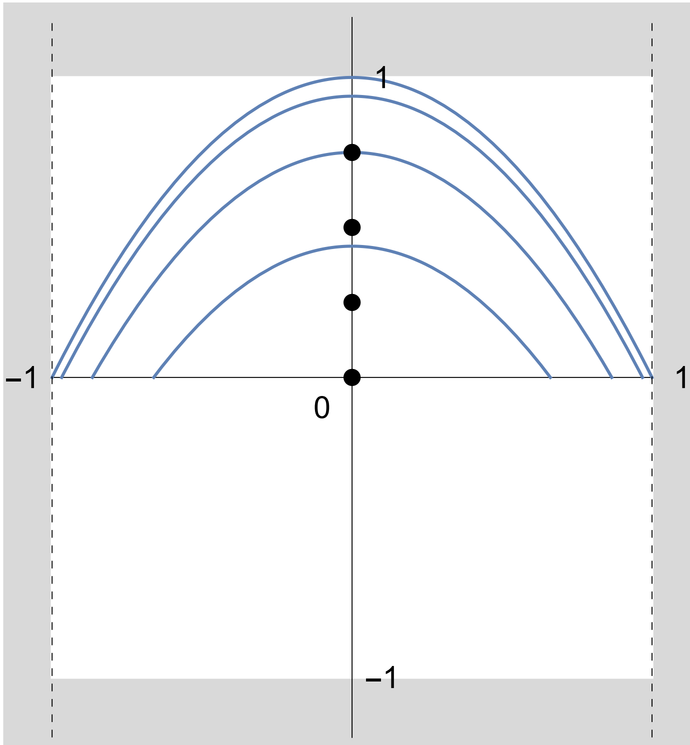

Figure 2. Parabolas $y+(k^2-1)=-x^2$,
$y \le 1$, $0 \le k < 1$ (k points on y axis), $|x| \le 1$