Needham, VDGF, Chapter 7, Exercise 14, p. 87.
G. Palmer, May 17, 2022, 5:20 pm PDT

Out[∘]=

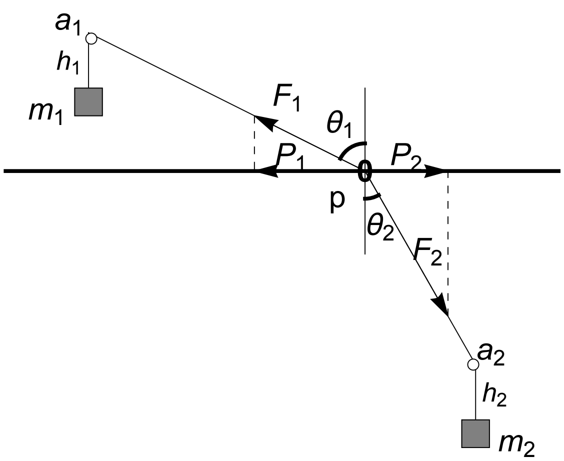

Figure 1. After VDGF, p. 87

14. Pólya's Mechanical Proof of Snell's Law.

The boundary between air and water is replaced by a frictionless rod, along which slides a ring, p. Attached to this ring are two strings of fixed length. As illustrated, each string passes over a frictionless peg at a_i and is attached to a mass m_i that hangs a distance h_i below a_i.

(i) By considering the potential energy of the system, deduce that equilibrium is achieved when

$$m_1 h_1 + m_2 h_2 = \text{maximum}$$

Let U be potential energy of the system. Gravity g is assumed to be constant. Then one can write

$$U = \Sigma_i m_i g (H_i - h_i)$$

where H_i is the height above the earth of one of the frictionless pegs. The quantities m_i, g, and H_i are all fixed.

$$U = \Sigma_i (m_i g H_i - m_i g h_i)$$

Since g is the same in both summands, we can say that when $m_1 h_1 + m_2 h_2$ = maximum, U = minimum. When U = minimum, the system is in equilibrium.

QED

(ii) But for equilibrium to occur, the horizontal components of the forces F_i pulling on the ring must cancel. Deduce that $m_1 \sin\theta_1 = m_2 \sin\theta_2$.

The force in the horizontal direction is the projection P_i of F_i onto the rod, which by p. 27, *VDG*, is

$$P = \mathbf{a} \bullet \mathbf{b} = |a||b|\cos\theta.$$

$$P_1 = |F_1| \; |-1| \cos(\pi/2 - \theta_1)$$

$$= |F_1| \cos(\pi/2 - \theta_1) = |F_1| \sin(\theta_1)$$

$$P_2 = |F_2||1| \cos(-\pi/2 + \theta_2) = |F_2| \cos(\pi/2 - \theta_2)$$

$$= |F_2| \sin(\theta_1)$$

The balance occurs when these two forces are equal.

$$|F_1| \sin\theta_1 = |F_2| \sin\theta_2$$

Since one can construe the exercise in terms of positive forces, $|F_1|$ may be written $m_i g$. Substitute for F_i and cancel g on both sides

$$m_1 \sin\theta_1 = m_2 \sin\theta_2$$

QED

(iii) Let l_i be the length of string from a_i to p, so that $l_i + h_i = const$. Now let us choose $m_i = (1/v_i)$, where v_1 and v_2 represent the speed of light in air and water, respectively, as in the original optical problem. Deduce that the mechanical problem in part (i) becomes mathematically identical to the original problem of finding the optical path of minimum time:

$$\frac{l_1}{v_1} + \frac{l_2}{v_2} = minimum$$

In the first sentence $l_i + h_i = const.$ suggests that the two strings have the same length, that $l_1 + h_1 = l_2 + h_2$, but that appears to contradict the diagram and the construction of [5.6]. Therefore, I will let $l_i + h_i = L_i$, where L_i is a constant.
Since $m_1 h_1 + m_2 h_2$ = maximum for these masses, one can write, given that $h_i = L_i - l_i$,

$$m_1(L_1 - l_1) + m_2(L_2 - l_2) = \text{maximum}$$

Since m_i and L_i are given, it follows that $m_1 l_1 + m_2 l_2$ = minimum.

Then substitute $m_i = (1/v_i)$ to obtain

$$\frac{l_1}{v_1} + \frac{l_2}{v_2} = \textit{minimum}$$

which is what we wanted.

I don't understand why one can simply let $m_i = (1/v_i)$.

(iv) From the solution of the mechanical problem in part (ii), deduce Snell's Law!

The solution from part (ii) is

$$m_1 \sin\theta_1 = m_2 \sin\theta_2$$

Substitute for m_i.

$$\frac{1}{v_1} \sin\theta_1 = \frac{1}{v_2} \sin\theta_2$$

$$\sin\theta_1 = \frac{v_1}{v_2} \sin\theta_2$$

$$\sin\theta_1 = n \sin\theta_2, \text{ where n = const. (p. 59)}$$

This is Snell's Law