Needham, VDGAF, Chapter 7, Exercise 13, pp. 86-87.
Palmer, April 19, 2022, 5:10 pm PDT

13. An Impossibly Good Map. ...Can we have a map that preserves both [area and angles]*? Show that no such map can exist for any curved surface, not just the sphere.*

The Local Gauss-Bonnet Theorem says the angular excess of a geodesic triangle on a general curved surface of variable curvature is expressed as (2.6), p. 23

$$\mathcal{E}(\triangle) = \iint_{\triangle} \mathcal{K} \, d\mathcal{A},$$

where $\mathcal{K}$ is the curvature of an infinitesimally small triangle and $\mathcal{A}$ is the area of the same triangle. Any shape on a general surface can be subdivided into geodesic triangles whose angular excess can be found with the integral. Since the angular excess of any subdivided triangle is additive on the parts, it follows that the angular excess of a shape composed of geodesic triangles is also additive. Since $\mathcal{K}$ may be positive or negative or 0 for any of the constituent triangles, it is theoretically possible that the angular excess over a surface may sum to 0, but in general, every curved surface has an angular excess that may vary over its constituent triangles. Hence, if a shape from an Euclidean surface is mapped onto a general curved surface in such a way that area is preserved, positive or negative angular excess on the surface reveals that angles are not preserved.