Needham, VDGAF, Chapter 7, Exercise 12, pp. 86-87.
Palmer, April 11, 2022, 5:25 pm PDT

12. Mercator Projection....In 1569 Gerardus Mercator discovered a third projection that [(1) mapped meridians of the sphere to vertical generators of the cylinder and (2) mapped circles of latitude to horizontal circular cross sections of the cylinder. The projection was also conformal].

Out[46]=

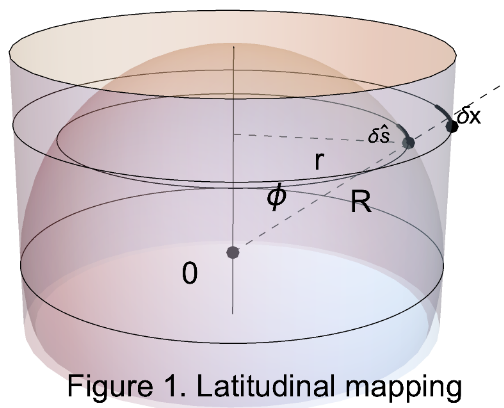
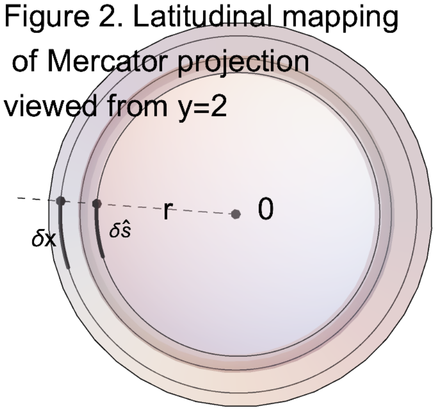


Figure 1. Latitudinal mapping of Mercator projection

Out[52]=

Figure 2. Latitudinal mapping of Mercator projection viewed from y=2



(i) Argue geometrically that if $\delta\hat{s}$ is a small movement along a circle of latitude on the sphere then the two required properties above imply that $\delta\,\hat{s} \simeq \sin\phi\;\delta s$, where $\delta s = \delta x$ is the horizontal movement on the cylinder.

The horizontal movement δx on the cylinder maps to a segment $\delta\hat{s}$ of a circle of latitude on the sphere. The radius of a circle of latitude can be defined as $r = R\sin\phi$ (See Figure 1). Then the movement on the circle of latitude can be written $\delta\,\hat{s} = r\,\delta\phi = R\sin\phi\,\delta\theta$. The corresponding horizontal movement $\delta s = \delta x$ on the cylinder can be written $\delta s = R\,\delta\theta$, so $\delta\theta = \frac{\delta s}{R}$. Then

$$\delta\,\hat{s} \simeq R\sin\phi\,\delta\theta$$

$$\asymp R \sin\phi \, \frac{\delta s}{R} \asymp \sin\phi \, \delta s$$

(ii) For our map to be conformal we must insist that we have the same scale factor for infinitesimal movements in all directions. In particular, suppose $\delta\hat{s}$ is instead chosen along a meridian, and let y = f(ϕ) be the height on the cylinder of the image of (θ, ϕ) on the sphere. Deduce that

$$f'(\phi) = -\frac{R}{\sin\phi}.$$

Note: The sign of the RHS is negative according to the Static Errata for VDGF.

Notice first that in figure 1, the line from the origin to the starting point of δx on the sphere does not touch the starting point of δs on the cylinder, the height of which was calculated with the equation for y in part (iii). It has the same theta-angle, but at this phi-angle, it hits the y-axis above the actual y value. A vertical movement δy on the cylinder maps to a vertical (meridianal) segment on the sphere, so δy corresponds to $-\delta$s. The meanings of δs and $\delta\hat{s}$ have shifted from horizontal to vertical. Since the movements on the sphere must have the same scale factor in all directions, we can again write $\delta\hat{s} = R\delta\phi$. From part (i),

$$\delta s \asymp \frac{\delta\hat{s}}{\sin\phi} \asymp \frac{R\,\delta\phi}{\sin\phi}$$

$$f'(\phi) = \frac{dy}{d\phi} = \frac{-\delta s}{\delta\phi} \asymp -\frac{R}{\sin\phi}$$

(iii) If we insist that points on the equator map to the x-axis, verify (or deduce, if you remember your integration techniques) that the Mercator projection is given by

$$y = f(\phi) = R \, ln\left[\frac{1+\cos\phi}{\sin\phi}\right]$$

My intent is to verify the equation by using differentiation on y to obtain $f'(\phi) = -\frac{R}{\sin\phi}$. Since dy decreases with increasing ϕ, we write $f'(\phi) = \frac{dy}{d\phi}$.

$$f'(\phi) = \frac{dy}{d\phi} = d(R\,[\ln(1+\cos\phi) - \ln(\sin\phi)])$$

$$= R\left[-\frac{\sin\phi}{1+\cos\phi} - \cot\phi\right] = -R\left[\frac{\sin\phi}{1+\cos\phi} + \frac{\cos\phi}{\sin\phi}\right]$$

$$= -R\,\frac{\sin^2\phi + \cos^2\phi + \cos\phi}{\sin\phi\,(1+\cos\phi)} = -R\,\frac{1+\cos\phi}{\sin\phi\,(1+\cos\phi)}$$

$$= -\frac{R}{\sin\phi} \qquad \text{QED}$$

(iv) A loxodromic route will appear to be a straight line on a chart, because travel on a fixed compass reading means travel at a constant angle with lines of latitude, which are parallel. That constant angle is preserved by the conformality of the projection to the cylinder, which when unrolled, becomes the chart.