Needham, VDGAF, Chapter 7, Exercise 11, pp. 85-86.

Palmer, March 31, 2022, 12:25 pm PDT

11. Central cylindrical Projection. Reconsider the cylinder and sphere in the diagram of the previous question, but now imagine that the cylinder extends infinitely upward and downward. We again project the sphere onto the cylinder, but this time by imagining a point source of light at the centre of the sphere, casting shadows onto the cylinder. This is called central cylindrical projection.

Out[3278]=

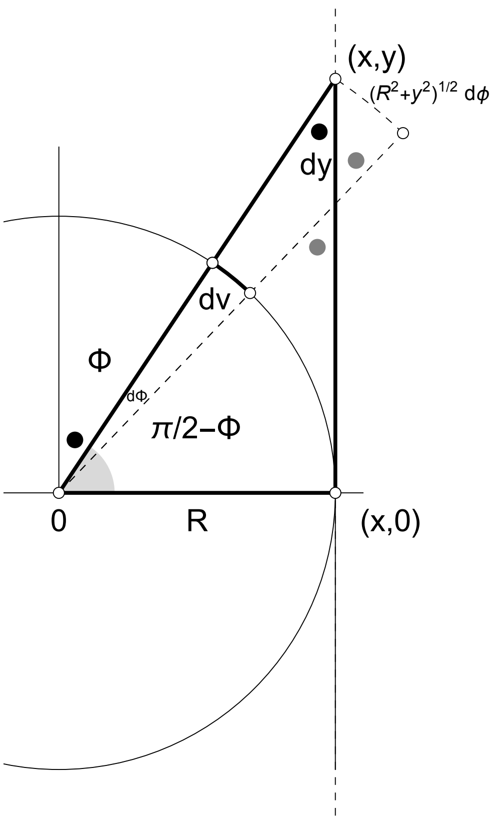


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Figure 1. Vertical cross section
of sphere and cylinder
through polar and y axes at any θ
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(i) Sketch the projection and deduce that $y = R \cot \phi$.

From Figure 1 and trig identities,

$$\tan(\pi/2 - \phi) = y/R = \cot \phi$$

Hence, $y = R \cot \phi$.

(ii) Either using part (i), or directly geometrically, show that the metric now takes the form

$$ds^2 = \frac{R^2}{R^2+y^2}\,dx^2 + \frac{R^4}{\left(R^2+y^2\right)^2}\,dy^2$$

In Exercise 10, the ratio du/dx is proportional to the distance from a point of origin at y on the polar axis of S to the point (u, v) on the surface of S and to the point (x, y) on the cylinder. In this exercise the rays originate at the origin, so du/dx is proportional to R/|(x,y)| (Figure 1). du and dx are not shown in the figure.

$$\frac{\mathrm{du}}{\mathrm{dx}} = \frac{R}{|(x,y)|} = \frac{R}{\sqrt{R^2+y^2}}$$

$$\mathrm{du} = \frac{R}{\sqrt{R^2+y^2}}\,\mathrm{dx}$$

$$\mathrm{du}^2 = \frac{R^2}{R^2+y^2}\,\mathrm{dx}^2$$

To find dv, we can write

$$\frac{\mathrm{dv}}{\mathrm{dy}} \asymp \frac{R^2}{|(x,y)|^2} = \frac{R^2}{R^2+y^2} \qquad (1)$$

My first attempt uses calculus. From (i) we have

$$\mathrm{y} = \mathrm{R}\cot\phi \quad \Longrightarrow \quad \cot\phi = \mathrm{y/R}$$

$$\partial_y \mathrm{ArcCot}[y/R] = -\frac{R}{R^2+y^2} \quad \text{(via Mathematica)}$$

$$\mathrm{dv} = R\left(-\frac{R}{R^2+y^2}\right)\mathrm{dy}$$

$$\mathrm{dv}^2 = \frac{R^4}{\left(R^2+y^2\right)^2}\,\mathrm{dy}^2$$

Then

$$d\hat{s}^2 = \mathrm{du}^2 + \mathrm{dv}^2 = \frac{R^2}{R^2+y^2}\,\mathrm{dx}^2 + \frac{R^4}{\left(R^2+y^2\right)^2}\,\mathrm{dy}^2$$

My geometric answer is adapted from Vasco’s answer. In Figure 1, a black corner ball indicates an angle of ϕ. A gray corner ball symbol indicates an angle ultimately equal to ϕ. Of course, right angles occur wherever an arc centred at the origin crosses a ray centred at the origin. As dϕ approaches 0, the small triangle on the upper right with hypotenuse dy and one side formed as the arc |(x,y)| dϕ is ultimately similar to the bold triangle with sides R and y and hypotenuse $\sqrt{R^2 + y^2}$. dv is the arc R dϕ, but we need to find dv in terms of R and y. We can write

$$\frac{\sqrt{R^2+y^2}\ \mathrm{d}\phi}{\mathrm{dy}} \asymp \frac{R}{\sqrt{R^2+y^2}}$$

$$\frac{\sqrt{R^2+y^2}\ (\mathrm{dv}/R)}{\mathrm{dy}} \asymp \frac{R}{\sqrt{R^2+y^2}} \quad \text{(substitute dv/R for d}\phi\text{)}$$

$$\mathrm{dv} = \frac{R^2}{R^2+y^2}\ \mathrm{dy}$$

The remainder of the proof is as in the version using calculus.

(iii) Apply the curvature formula (4.10) to confirm that $\mathcal{K} = +1/R^2$.

$$A = \frac{\mathrm{du}}{\mathrm{dx}} = \frac{R}{\sqrt{R^2+y^2}}$$

$$B = \frac{\mathrm{dv}}{\mathrm{dy}} = \frac{R^2}{R^2+y^2}$$

$$\frac{1}{A\,B} = \frac{\left(R^2+y^2\right)^{3/2}}{R^3}$$

$$\frac{\partial_y A}{B} = -\frac{y}{R\ \sqrt{R^2+y^2}}$$

$$\frac{\partial_x B}{A} = 0$$

$$\partial_y\left[-\frac{y}{R\ \sqrt{R^2+y^2}}\right] = -\frac{R}{\left(R^2+y^2\right)^{3/2}}$$

$$\mathcal{K} = -\frac{\left(R^2+y^2\right)^{3/2}}{R^3}\left(\partial_y\left[-\frac{y}{R\ \sqrt{R^2+y^2}}\right] + \partial_x[0]\right)$$

$$= -\frac{\left(R^2+y^2\right)^{3/2}}{R^3}\left(-\frac{R}{\left(R^2+y^2\right)^{3/2}}\right)$$

$$= +\frac{1}{R^2} \qquad \text{QED}$$