Needham, VDGAF, Chapter 7, Exercise 10, pp. 85-86.
Palmer, March 28, 2022, 12:30 pm PDT

Out[1646]=

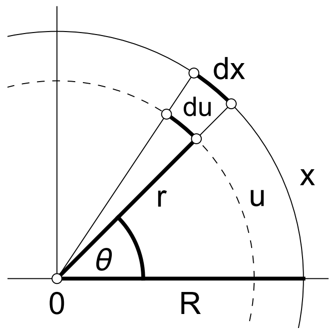

Figure 1. Horizontal cross section of sphere parallel to equator at y

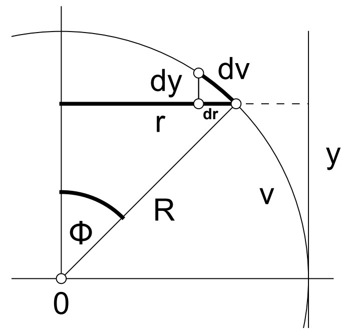

Figure 2. Vertical cross section of sphere through y axis at x

10. Archimedes-Lambert Projection. In the (x, y)-plane, consider the rectangle ($0 \leq x \leq 2\pi R$, $-R \leq y \leq R$). Now imagine first taping together the left and right edges to create a cylinder, then slipping this over a sphere of radius R so that the cylinder touches the sphere along its equator, ... We now draw a map of the sphere by projecting its points horizontally onto the cylinder, radially outward, perpendicular to the axis of the cylinder, ... This is the Archimedes-Lambert projection....

(i) Show (ideally geometrically) that the metric of the sphere now takes the form

$$d\hat{s}^2 = \left(R^2 - y^2\right) dx^2 + \frac{dy^2}{R^2 - y^2}$$

For simplicity and ease of typesetting, I use only the infinitesimals and the asymptotic "≍" symbol. Coordinates (x, y) refer to the cylinder and (u, v) refer to the surface of the sphere. Since the rectangle wraps around the sphere, du $\asymp \frac{r}{R}$ dx as shown in Figure 1, where $R^2 = y^2 + r^2$ (Figure 2).

$$\mathrm{du} \asymp \frac{\sqrt{R^2 - y^2}}{R}\,\mathrm{dx}$$

$$\mathrm{du}^2 \asymp \frac{R^2 - y^2}{R^2}\,\mathrm{dx}^2$$

From Figure 2, one can deduce that $\mathrm{dv}^2 \asymp \mathrm{dy}^2 + \mathrm{dr}^2$ and $R^2 = y^2 + r^2$. Then

$$r^2 = R^2 - y^2 \Longrightarrow 2\,\mathrm{rdr} = -2\,\mathrm{ydy}$$

$$\mathrm{dr} = \frac{-\mathrm{ydy}}{r} = \frac{-\mathrm{ydy}}{\sqrt{R^2-y^2}} \Longrightarrow \mathrm{dr}^2 = \frac{y^2\,\mathrm{dy}^2}{R^2-y^2}$$

$$\mathrm{dv}^2 \asymp \mathrm{dy}^2 + \mathrm{dr}^2 = \mathrm{dy}^2 + \frac{y^2\,\mathrm{dy}^2}{R^2-y^2} = \frac{R^2\,\mathrm{dy}^2}{R^2-y^2}$$

$$d\hat{s}^2 = \mathrm{du}^2 + \mathrm{dv}^2 \asymp \frac{R^2-y^2}{R^2}\,\mathrm{dx}^2 + \frac{R^2}{R^2-y^2}\,\mathrm{dy}^2$$

This is not what Needham has, but Vasco has pointed out that Needham's equation cannot be correct because it has an area dimension with power of 2 on the LHS and and powers of 4 and 0 on the RHS.

My answer uses calculus at a crucial point, as Vasco has pointed out (PC, VDGAF forum). The more "ideal" approach is what I think of as Needham's Newtonian geometrical approach. Since θ is equal for both x and u, $\mathrm{dx} = \mathrm{d}\theta\,\mathrm{R}$, $\mathrm{du} = \mathrm{d}\theta\,\mathrm{r}$, and one can write an equation for the ratios:

$$\mathrm{d}\theta = \frac{\mathrm{dx}}{R} = \frac{\mathrm{du}}{r}$$

$$\mathrm{du} = \frac{r}{R}\,\mathrm{dx} = \frac{\sqrt{R^2-y^2}}{R}\,\mathrm{dx}$$

Now find dv, given that in Figure 2, the small triangle with hypotenuse dv is ultimately similar to the large triangle with hypotenuse R and base r. It is not precisely similar because dv is is a segment of a geodesic curve, not an euclidean straight line.

$$\frac{\mathrm{dy}}{\mathrm{dv}} \asymp \frac{r}{R} = \frac{\sqrt{R^2-y^2}}{R}$$

$$\mathrm{dv} \asymp \frac{R}{\sqrt{R^2-y^2}}\,\mathrm{dy}$$

$$d\hat{s}^2 = \mathrm{du}^2 + \mathrm{dv}^2 \asymp \frac{R^2-y^2}{R^2}\,\mathrm{dx}^2 + \frac{R^2}{R^2-y^2}\,\mathrm{dy}^2$$

(ii) Apply the curvature formula (4.10) to show that $\mathcal{K} = +\frac{1}{R^2}$.

Set u = x, v = y.

From part (i),

$$\mathrm{du} \asymp \frac{\sqrt{R^2-y^2}}{R}\,\mathrm{dx} \implies A = \frac{\partial u}{\partial x} \asymp \frac{\sqrt{R^2-y^2}}{R}$$

$$\mathrm{du}^2 \asymp \frac{R^2\,\mathrm{dy}^2}{R^2-y^2}$$

$$\mathrm{dv} \asymp \frac{R}{\sqrt{R^2-y^2}}\,\mathrm{dy} \implies B \asymp \frac{\partial v}{\partial y} = \frac{R}{\sqrt{R^2-y^2}}$$

$$\mathcal{K} = -\frac{1}{\mathrm{AB}}\left(\partial_y\left[\frac{\partial_Y A}{B}\right] + \partial_x\left[\frac{\partial_x B}{A}\right]\right) \qquad (4.10)$$

$$\mathrm{AB} = \frac{\sqrt{R^2-y^2}}{R}\cdot\frac{R}{\sqrt{R^2-y^2}} = 1$$

$$\partial_v A = \partial_y\left[\frac{\sqrt{R^2-y^2}}{R}\right] = -\frac{y}{R\sqrt{R^2-y^2}}, \qquad \partial_x B = \partial_x\left[\frac{R}{\sqrt{R^2-y^2}}\right] = 0$$

$$\mathcal{K} = -\partial_y\left[\frac{-\frac{y}{R\sqrt{R^2-y^2}}}{\frac{R}{\sqrt{R^2-y^2}}}\right] + \partial_x\left[\frac{0}{A}\right] = -\partial_y\left[-\frac{y}{R^2}\right] = +\frac{1}{R^2}$$

(iii) Use (4.12) to deduce that the projection preserves area. For example, the two illustrated T-shapes have equal area....

It is given by construction that dx ⊥ dy and du ⊥ dv. Then it should suffice to show that du dv ≍ dx dy.

$$\mathrm{du\,dv} \asymp \frac{\sqrt{R^2-y^2}}{R}\,\mathrm{dx}\,\frac{R}{\sqrt{R^2-y^2}}\,\mathrm{dy} = \mathrm{dxdy}$$

(iv) Use the previous part to deduce (without integration) that the area $\mathcal{A}$ of the polar cap $0 \leq \phi \leq \Phi$ is

$$\mathcal{A} = 2\,\pi \mathrm{R}^2(1 - \cos\Phi).$$

The symbol ϕ has been given different interpretations in previous chapters. It was introduced in (2.4), p. 19 as the angle between the Z-axis and the equator of the sphere. That is the interpretation used here. We take capital Φ to be an arbitrary fixed lower boundary of a polar cap, where $0 \leq \Phi \leq \pi/2$. Φ is the angle at the origin between the polar axis and the lower boundary of the cap. Lowercase ϕ is a variable that is not used in this calculation.

From (iii) the projection preserves area. The polar cap is the projection to the sphere from a segment of the cylinder having circumference 2π R and length (height) of $(R - R\cos\Phi)$ (Figure 2). The area of the polar cap is the product of these, and it is as in the exercise statement.

$$\mathcal{A} = 2\,\pi\,R\,(\,R - R\cos\Phi) = 2\,\pi\,R^2\,(1 - \cos\Phi)$$