Needham, VDGAF, Chapter 7, Exercise 1, p. 83
Palmer, April 25, 2022, 3 pm PST

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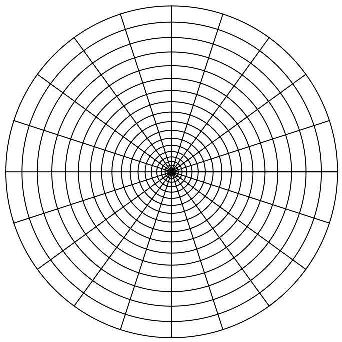

$x = u^2 \cos v$
$y = u^2 \sin v$

*1. Coordinate-Independence of $\mathcal{K}$. Begin with the flat Euclidean metric,*

$$d\hat{s}^2 = dx^2 + dy^2$$

*(i) If $x = u^2 \cos v$ and $y = u^2 \sin v$, interpret the (u, v) coordinates geometrically and deduce that they are orthogonal.*

The v coordinate determines traversal around a circle (r cos v, r sin v). The u coordinate determines r, the radius of the circle, where $r = u^2$ at a given angle of rotation v. At a given v, variation on u produces a ray. We can think of the (u, v) coordinates as v-rays emanating from a centre q and u-circles centred on q (Figure). Hence, we can deduce that the (u, v) coordinates are orthogonal.

*(ii) Verify this orthogonality by calculating the metric in the (u, v) coordinates, and apply the curvature formula (4.10) to confirm that this new metric formula is still flat, as it should be: $\mathcal{K} = 0$.*

Since the coordinates are orthogonal rays and circles, the metric can be described with Euclidean coordinates: $d\hat{s}^2 = \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2$. Then the equation for $\mathcal{K}$ given in paragraph 3, p. 38 applies. Note that $r = u^2$ and $\theta = v$, so that dr = 2udu and $r\, \mathrm{d}\theta = u^2\, \mathrm{d}v$. Then writing the metric in terms of (u, v),

$$d\hat{s}^2 = 4\, u^2\, \mathrm{d}u^2 + u^4\, \mathrm{d}v^2$$

The partial derivatives can be used in (4.10) to calculate $\mathcal{K}$.

$$A = \frac{\partial r}{\partial u} = \frac{\partial (u^2)}{\partial u} = 2\, u \qquad \Longrightarrow \qquad \frac{\partial_v A}{B} = 0 \quad \text{and} \quad \partial_v \frac{\partial_v A}{B} = 0$$

$$B = \frac{\partial \theta}{\partial v} = \frac{\partial v}{\partial v} = 1 \qquad \Longrightarrow \qquad \frac{\partial_u B}{A} = 0 \quad \text{and} \quad \partial_u \frac{\partial_u B}{A} = 0$$

Then without further calculation, we can say that $\mathcal{K} = 0$.

Figure 2

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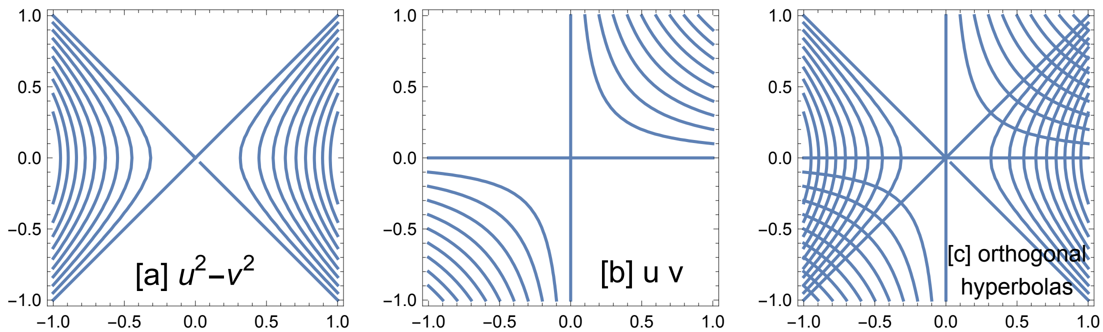


*(iii) Changing coordinates again, show that the orthogonal trajectories of the hyperbolas $x = u^2 - v^2$ = const. are the hyperbolas $y = 2uv$ = const. Repeat (ii) for these conformal coordinates, using (4.10) or (4.16) to confirm that $\mathcal{K} = 0$.*

*The first part is answered by showing that the derivatives of the hyperbolas x and y are orthogonal at all points (x, y) on the curves.*

The derivatives of x and y with respect to u are orthogonal at all points on the curves: $\frac{dx}{du} = 2u$, $\frac{dy}{du} = 2v$

The derivatives of x and y with respect to v are orthogonal at all points on the curve: $\frac{dx}{dv} = -2v$, $\frac{dy}{dv} = 2u$

The derivatives of x with respect to u and v are orthogonal at all points on the curve: $\frac{dx}{du} = 2u$, $\frac{dx}{dv} = -2v$

The derivatives of y with respect to u and v are orthogonal at all points on the curve: $\frac{dy}{du} = 2v$, $\frac{dy}{dv} = 2u$

In every case the trajectories of the derivatives are orthogonal (assuming that u and v are orthogonal).

*Repeat (ii) for these conformal coordinates, using (4.10) or (4.16) to confirm that $\mathcal{K} = 0$: (iii) Verify this orthogonality by calculating the metric in the (u, v) coordinates, and apply the curvature formula (4.10) to confirm that this new metric formula is still flat, as it should be: $\mathcal{K} = 0$.*

$$dx = \partial_u x \, du + \partial_v x \, dv = 2u \, du - 2v \, dv$$
$$dy = \partial_u y \, du + \partial_v y \, dv = 2v \, du + 2u \, dv$$

$$d\hat{s}^2 = \mathrm{dx}^2 + \mathrm{dy}^2 = (2\,u\,\mathrm{du}\; - 2\,v\,\mathrm{dv})^2 + (2\,v\,\mathrm{du} + 2\,u\,\mathrm{dv})^2$$

$$= \left(4\,u^2\,\mathrm{du}^2 - \;4\,\mathrm{uv}\,\mathrm{dudv} + 4\,v^2\,\mathrm{dv}^2\right) + \left(4\,v^2\,\mathrm{du}^2 + 4\,\mathrm{uvdudv} + 4\,u^2\,\mathrm{dv}^2\right)$$

$$= \left(4\,u^2\,\mathrm{du}^2 + 4\,v^2\,\mathrm{dv}^2\right) + \left(4\,v^2\,\mathrm{du}^2 + 4\,u^2\,\mathrm{dv}^2\right)$$

$$= 4\left(u^2 + v^2\right)\mathrm{du}^2 + 4\left(u^2 + v^2\right)\mathrm{dv}^2 = 4\left(u^2 + v^2\right)\left(\mathrm{du}^2 + \mathrm{dv}^2\right)$$

Using (4.16) $\mathcal{K} = -\frac{\nabla^2 \ln\Lambda}{\Lambda^2}$

$$d\hat{s}^2 = 4\left(u^2 + v^2\right)\mathrm{du}^2 + 4\left(u^2 + v^2\right)\mathrm{dv}^2$$

$$d\hat{s}^2 = \mathrm{dr}^2 + r^2\,\mathrm{d}\theta^2$$

$$\mathrm{dr}^2 = 4\left(u^2 + v^2\right)\mathrm{du}^2$$

$$A = \frac{\mathrm{dr}}{\mathrm{du}} = 2\left(u^2 + v^2\right)^{1/2}$$

$$\Lambda = \mathrm{B} = \mathrm{A}$$

$$\partial_u \ln\Lambda = \frac{u}{u^2+v^2}$$

$$\partial_v \ln\Lambda = \frac{v}{u^2+v^2}$$

$$\partial_u \partial_u \ln\Lambda = \frac{-u^2+v^2}{\left(u^2+v^2\right)^2}$$

$$\partial_v \partial_v \ln\Lambda = \frac{u^2-v^2}{\left(u^2+v^2\right)^2}$$

$$\mathcal{K} \;=\; -\frac{1}{4\left(u^2+v^2\right)}\left[\frac{-u^2+v^2}{\left(u^2+v^2\right)^2} + \frac{u^2-v^2}{\left(u^2+v^2\right)^2}\right] = 0$$