Notes for Needham, Visual Differential Geometry and Forms, Chapter 6
G. Palmer, October 13, 2022

P. 72. *The matrix results (6.8) and (6.9) makes it a simple matter [exercise] to verify that each of these three sets is indeed a group.*

Set 1. $\mathcal{K} = 0$. $E(z) = e^{i\theta} z + k$

Rewrite as $[E_1(z)] = \begin{bmatrix} e^{i\theta} & k \\ 0 & 1 \end{bmatrix}$ and let $[E_2(z)] = \begin{bmatrix} e^{i\phi} & m \\ 0 & 1 \end{bmatrix}$.

Composition: Show that $[E_2 \circ E_1] = [E_2][E_1]$

$$[E_2 \circ E_1] = \left[\left(e^{i\phi} z + m\right) \circ \left(e^{i\theta} z + k\right)\right] = \left[e^{i\phi}\left(e^{i\theta} z + k\right) + m\right]$$

$$\left[e^{i(\theta+\phi)} z + e^{i\phi} k + m\right] \equiv \begin{bmatrix} e^{i(\theta+\phi)} & ke^{i\phi} + m \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\phi} & m \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{i\theta} & k \\ 0 & 1 \end{bmatrix} = [E_2][E_1]$$

Inverse: Show that $\left[E^{-1}\right] = [E]^{-1}$

From p. 150, *VCA*, if $E = \frac{az+b}{cz+d} \to \frac{e^{i\theta} z+k}{1}$ and $(ad - bc) \neq 0$, then

$$E^{-1} = \frac{dz-b}{-cz+a} \to \frac{z-k}{e^{i\theta}}$$

$$\left[E^{-1}\right] = \begin{bmatrix} 1 & -k \\ 0 & e^{i\theta} \end{bmatrix}$$

Then

$$\left[E^{-1}\right] \circ [E] = \begin{bmatrix} 1 & -k \\ 0 & e^{i\theta} \end{bmatrix} \cdot \begin{bmatrix} e^{i\theta} & k \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\theta} & k - k \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e \text{ (identity)}$$

The association test follows from the composition test. The composition and inverse tests therefore show that $E \in \mathcal{G}(\mathcal{E})$.

Set 2. $\mathcal{K} = +1$. $S(z) = \frac{az+b}{-\bar{b}\,z+\bar{a}}$, where $|\,a\,|^2 + |\,b\,|^2 = 1$

Composition: Show that $[S_2 \circ S_1] = [S_2][S_1]$.

Let

$$S_1(z) = \frac{az+b}{-\bar{b}\,z+\bar{a}} \equiv \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = [S_1]$$

then

$$S_2(z) = \frac{cz+d}{-\bar{d}\,z+\bar{c}} \equiv \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} = [S_2]$$

$$S_2 \circ S_1 = \frac{c\,\frac{az+b}{-\bar{b}z+\bar{a}}+d}{-\bar{d}\,\frac{az+b}{-\bar{b}z+\bar{a}}+\bar{c}} = \frac{\left(ac-\bar{b}\,d\right)z+\bar{a}\,d+bc}{\left(-a\,\bar{d}-\overline{bc}\right)z+\overline{ac}-b\,\bar{d}}$$

$$[S_2 \circ S_1] = \begin{bmatrix} ac-\bar{b}\,d & \bar{a}\,d+bc \\ -a\,\bar{d}-\overline{bc} & \overline{ac}-b\,\bar{d} \end{bmatrix} = \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = [S_2][S_1]$$

Inverse: Show that $\left[S^{-1}\right] = [S]^{-1}$

Let $S = \frac{az+b}{-\bar{b}\,z+\bar{a}} \equiv \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = [S]$ and $(ad-bc) \neq 0$, and

$$S^{-1} = \frac{\bar{a}\,z-b}{\bar{b}\,z+a} \equiv \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix} = \left[S^{-1}\right]$$

Then

$$\left[S^{-1}\right]\circ[S] = \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}.\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

$$= \begin{bmatrix} |\,a\,|^2+|\,b\,|^2 & \bar{a}\,b-\bar{a}\,b \\ a\,\bar{b}-a\,\bar{b} & |\,a\,|^2+|\,b\,|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e \text{ (identity)}$$

The composition and inverse tests show that $S \in \mathcal{G}(\mathcal{S})$.

Set 3. $\mathcal{K} = -1$. $H(z) = \frac{az+b}{cz+d}$, where a, b, c, d are real, (ad − bc) = 1

Composition: Show that $[H_2 \circ H_1] = [H_2][H_1]$

Let $H_1(z) = H(z)$ as above, and let

$$H_1(z) = \frac{az+b}{cz+d} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [H_1]$$

and let

$$H_2(z) = \frac{ez-f}{gz+h} \equiv \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$H_2 \circ H_1 = \frac{e\,\frac{az+b}{cz+d}+f}{g\,\frac{az+b}{cz+d}+h} = \frac{(ae+cf)\,z+be+df}{(ag+ch)\,z+bg+dh}$$

$$[H_2 \circ H_1] = \begin{bmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [H_2][H_1]$$

Inverse: Show that $\left[H^{-1}\right] = [H]^{-1}$

Let $H = \frac{az+b}{cz+d} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [H]$ and $(ad - bc) \neq 0$, and

$$H^{-1} = \frac{dz-b}{-cz+a} \equiv \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \left[H^{-1}\right]$$

Then

$$\left[H^{-1}\right] \circ [H] = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e \text{ (identity)}$$

The composition and inverse tests show that $H \in \mathcal{G}(\mathcal{H})$.

[Exercise] A unitary matrix: $(\overline{[S]})^t$ (designated [S]*) is the inverse of a matrix of a spherical transformation. Show that $[S][S]^* = e$.

$$[S]^* = \text{transpose}\left(\text{conjugate}\left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\right)\right) = \text{transpose}\left(\begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix}\right) = \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}$$

$$[S]\,\overline{[S]}^t = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 & -\text{ab} + \text{ab} \\ -\overline{\text{ab}} + \overline{\text{ab}} & |a|^2 + |b|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e$$

P. 73. Direct calculation shows [exercise] that

$$1 + |w|^2 = \frac{1+|z|^2}{|-\bar{b}z+\bar{a}|^2}$$

From (6.12), $M'(z) = \frac{1}{(\text{cz}+d)^2}$, so substituting from (6.10), $S'(z) = \frac{1}{(-\bar{b}z+\bar{a})^2}$. One could also calculate this from $S = \frac{\text{az}+b}{-\bar{b}z+\bar{a}}$. By Leibnitz's quotient derivative, and noting that for S(z), $|a|^2 + |b|^2 = 1$,

$$\left(\frac{\text{az}+b}{-\bar{b}z+\bar{a}}\right)' = \frac{a(-\bar{b}z+\bar{a})-(-\bar{b})(\text{az}+b)}{(-\bar{b}z+\bar{a})^2} = \frac{-\text{abz}+\text{abz}+|a|^2+|b|^2}{(-\bar{b}z+\bar{a})^2} = \frac{1}{(-\bar{b}z+\bar{a})^2}$$

(6.13) postulates that the equations for $d\hat{s}$ are the same whether written in terms of w or of z. That is, the map in ℂ is isometric to the Riemann sphere.

$$\frac{2}{1+|w|^2}\,|\text{dw}| = \frac{2}{1+|z|^2}\,|\text{dz}| \Longrightarrow \frac{|\text{dw}|}{|\text{dz}|} = \left|\frac{\text{dw}}{\text{dz}}\right| = \frac{1+|w|^2}{1+|z|^2}$$

But this needs to be verified. By substituting 1 for R in (4.22), we get

$$\text{d}\hat{\text{s}} = \frac{2}{1+|z|^2}|\text{dz}|$$

By construction, dŝ =dw and |w|=R=1. Then, $\text{d}\hat{\text{s}} = \frac{2}{1+|w|^2}\,|\text{dw}|$ is equivalent to dŝ =|dw|. Therefore, $\text{d}\hat{\text{s}} = \frac{2}{1+|w|^2}|\text{dw}| = \frac{2}{1+|z|^2}|\text{dz}|$. This confirms (6.13), and it implies that

$$\left|\frac{\text{dw}}{\text{dz}}\right| = \frac{1+|w|^2}{1+|z|^2}$$

Since $|S'(z)| = \left|\frac{\text{dw}}{\text{dz}}\right|$,

$$\mid S'(z) \mid = \frac{1}{\left|-\overline{b}\, z+\overline{a}\right|^2} = \frac{1+|w|^2}{1+|z|^2}$$

I do not see how this confirms part 2 of the Main Result. It seems rather to provide an alternative way of writing |S'(z)|.

P. 73 A simple argument to validate the spherical transformation (6.10)

The text stipulates that *stereographic* images are related by $M\left(-\frac{1}{\overline{z}}\right) = -\frac{1}{\overline{M(z)}}$. The first equation writes M as the *general* Möbius transformation $\frac{az+b}{cz+d}$.

$$M\left(-\frac{1}{\overline{z}}\right) = \frac{a\left[-\frac{1}{\overline{z}}\right]+b}{c\left[-\frac{1}{\overline{z}}\right]+d} = \frac{-a+b\,\overline{z}}{-c+d\,\overline{z}} \qquad (1)$$

$$-\frac{1}{\overline{M(z)}} = -1/\overline{\left(\frac{az+b}{cz+d}\right)} = -\frac{\overline{c}\,\overline{z}+\overline{d}}{\overline{a}\,\overline{z}+\overline{b}} \qquad (2)$$

(1) and (2) are not obviously equal, but let $c = -\overline{b}$ and $d = \overline{a}$. Then substitute into (1) and (2) obtain $\frac{-a+b\,\overline{z}}{\overline{b}+\overline{a}\,\overline{z}} = -\frac{-b\,\overline{z}+a}{\overline{a}\,\overline{z}+\overline{b}}$, which is correct. So $M\left(-\frac{1}{\overline{z}}\right) = -\frac{1}{\overline{M(z)}}$ under the constraints $c = -\overline{b}$ and $d = \overline{a}$. By making the same substitutions into the general Möbius transformation $\frac{az+b}{cz+d}$ we obtain $\frac{az+b}{-\overline{b}\,z+\overline{a}}$, which is (6.10), the most general transformation for stereographic images in spherical geometry. But it might be more straightforward to perform the test directly on S(z), i.e. Let M = S (i.e. 6.10).

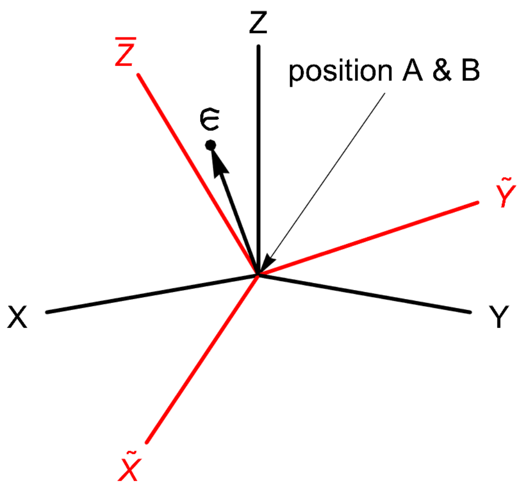
$$S\left(-\frac{1}{\overline{z}}\right) = \frac{a\left[-\frac{1}{\overline{z}}\right]+b}{-\overline{b}\left[-\frac{1}{\overline{z}}\right]+\overline{a}} = \frac{-a+b\,\overline{z}}{\overline{b}+\overline{a}\,\overline{z}} \qquad (3)$$

$$-\frac{1}{\overline{S(z)}} = -1/\overline{\left(\frac{az+b}{-\overline{b}\,z+\overline{a}}\right)} = -\frac{-b\,\overline{z}+a}{\overline{a}\,\overline{z}+\overline{b}} \qquad (4)$$

Since (3) = (4), (6.10) is valid.

Pp. 74-76. Notes on 6.4 Einstein's Spacetime Geometry.

Out[ ]=




```
Figure 1. Two Cartesian coordinate frames with event ∈
and observers A and B.
```

The construction presented in these pages has two observers, which I will call A and B. Needham wrote: *Let us combine the time T and the 3-dimensional Cartesian coordinates (X, Y, Z) of an* ***event ∈*** *into a single* ***4-vector*** *(T, X, Y, Z) in 4-dimensional* ***spacetime.*** *These are the space and time coordinates of event ∈ for what we shall call our first observer.*

I think what Needham means by this is that (T, X, Y, Z) is how observer A would describe the position of event ∈.

Note that T cannot simply be time in seconds or minutes, as equations using it show that it must have the same dimension as X, Y, and Z. T has this dimension because it is actually tc, time multiplied by c, the speed of light, which Einstein set (normalized) at 1 for convenience. Speed has units of distance over time, such as meters/second, so T must be a distance like X, Y, and Z, but it is relative to the Cartesian distances, so T, X, Y, and Z are all components of 4-dimensional spacetime. This is true also of intervals $\mathfrak{I}^2 = T^2 - \left(X^2 + Y^2 + Z^2\right)$, which are discussed in this section. Needham points out that *the speed of light is the same for all observers* ***in uniform relative motion*** (box, p. 84, emphasis added).

Observer B's vector is described with rotated coordinates illustrated in red in Figure 1, but the vector *uses the same origin* (p. 74, ln. -1). A and B are not in relative motion. They are in the same position, with the only difference being the rotation of their coordinates with respect to one another. They perceive event ∈ as originating at the same point at a distance from them. With this construction, one can write

$$\tilde{X}^2 + \tilde{Y}^2 + \tilde{Z}^2 = X^2 + Y^2 + Z^2$$

which shows that A and B agree on the distance to ∈ (p. 75, ln. 2).

On p. 75, ¶ 1, ln. 3, the situation seems to change. We learn that A and B are in relative motion, but "momentarily coincident". Even when they have the same locus, they disagree about the time at which events occur and disagree about the value of $X^2 + Y^2 + Z^2$! The effect of relative motion is called the *Lorenz contraction*. It is not clear whether the coordinates of B are still rotated with respect to those of A. Perhaps it doesn't matter.

On p. 75, ¶ 5, ln. 2, we learn that even though A and B are in relative motion and disagree about the value of $(X^2 + Y^2 + Z^2)$, they agree on the spacetime interval "ϒ" between observer and event.

$$ϒ^2 = T^2 - (X^2 + Y^2 + Z^2) = \tilde{T}^2 - (\tilde{X}^2 + \tilde{Y}^2 + \tilde{Z}^2) \qquad (6.15)$$

Out[•]=

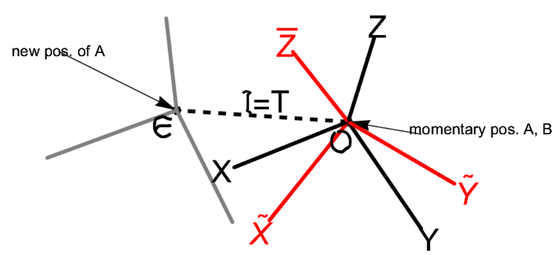


Figure 2. Two Cartesian coordinate frames
with events ○ and ∈ and observers A and B.
A uses black frame; B uses red.

On p. 75, ¶ -2, ln. -1, we find that the square of the *interval* between *distinct* events can be 0! On p. 75, ¶ -1, lns. 2 & 3, Needham describes two observers as momentarily coincident and in relative motion, *defining an origin event that we label* O (Figure 2) . The referent is vague. Evidently, it is the coincidence of the observers that defines O as an event emanating from their position. We know this because the observers are defined to be *pointlike, having a specific location*. On p. 75, ¶ -1, we learn that observer A arrives at an event ∈, apart from O, *just as it happens* and that ϒ ≡ T for A (Figure 2). ϒ is the **invariant** spacetime interval between ∈ and O. All observers must agree on this. ϒ is the elapsed time on A's watch (6.16). So it appears that A traversed spacetime from a position where he/she observed O and then moved on to the position where she observed ∈. But as Needham noted, she might perceive her own position to be stationary.

Out[•]=

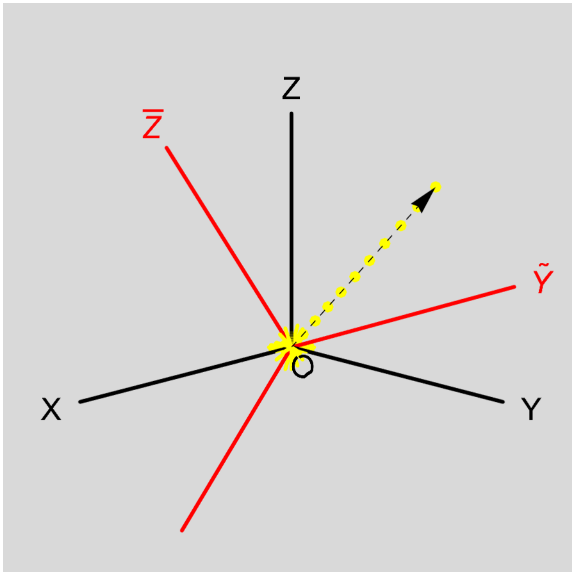


Figure 3. Spark at O with traversal of single photon

Needham proposes that a spark occurs at O (p. 76, ¶ 1). (But does he mean “occurred,” as A has moved on from O.) Remember, O is the event defined by the **momentary** coincidence of A and B. Now I think it is critical to know whether the construction includes the previously mentioned arrival of A at event Є, or whether we are starting over at O. I will assume that we are starting over at O and that A has not yet experienced Є. Both observers focus on a single photon, observing every event along the photon's spacetime trajectory. Both agree that every event on the trajectory has ST interval $\iota = 0$. Can one deduce that the photon is moving away from A and B along a ray? Suppose the photon were on the X axis of A’s frame. At a distance of 1, $\iota^2 = 1^2 - (1^2 + 0 + 0) = 0$. In B’s frame, the elapsed time would be different, because B is moving relative to A, and the distances would be different, but the squares would still sum to zero. Needham wrote (p. 76, ¶ 1, lns. 2 & 3) that this is because the vector has *vanishing length pointing along a particular light ray* (null vector). Why would it have vanishing length rather than increasing length? Furthermore, the vanishing length seems to contradict the construction in which A and B are in relative motion. What became of the Lorenz Contraction? How can A and B simultaneously (a) be in relative motion along unspecified rays from O, (b) both view successive events occurring along one ray of one photon, and (c) perceive no intervals between events even while time is elapsing for both? Is it that they do not perceive the spacetime interval? They must perceive changes in distance in order to measure them.

P. 76. *Substituting the projective-coordinate description* $z = (\zeta_1 / \zeta_2)$ *into the stereographic formulas (4.25)* [p. 48]*, we obtain,*

$$X = \frac{\zeta_1 \overline{\zeta_2} + \zeta_2 \overline{\zeta_1}}{|\zeta_1|^2 + |\zeta_2|^2},\ Y = \frac{\zeta_1 \overline{\zeta_2} - \zeta_2 \overline{\zeta_1}}{i\left(|\zeta_1|^2 + |\zeta_2|^2\right)},\ \text{and } Z = \frac{|\zeta_1|^2 - |\zeta_2|^2}{|\zeta_1|^2 + |\zeta_2|^2}$$

Substitute into

$$X + \mathrm{i}Y = \frac{2z}{1+|z|^2} \text{ and } Z = \frac{|z|^2-1}{|z|^2+1} \qquad \text{(from 4.25)}$$

Write X and Y using Re(z) = $\frac{1}{2}(z+\overline{z})$ and Im(z) = $\frac{1}{2i}(z-\overline{z})$ from p. 7, *VCA*.

$$X = \frac{1}{2}\left(\frac{2z}{1+|z|^2} + \frac{2\overline{z}}{1+|z|^2}\right) = \frac{(z+\overline{z})}{1+|z|^2} = \frac{\left(\zeta_1/\zeta_2+\overline{\zeta_1}/\overline{\zeta_2}\right)}{1+\frac{|\zeta_1|^2}{|\zeta_2|^2}} = \frac{\left(\zeta_1/\zeta_2+\overline{\zeta_1}/\overline{\zeta_2}\right)\left(\zeta_2\,\overline{\zeta_2}\right)}{|\zeta_2|^2+|\zeta_1|^2} = \frac{\zeta_1\,\overline{\zeta_2}+\zeta_2\,\overline{\zeta_1}}{|\zeta_2|^2+|\zeta_1|^2}$$

$$Y = \frac{1}{2i}\left(\frac{2z}{1+|z|^2} - \frac{2\overline{z}}{1+|z|^2}\right) = \frac{(z-\overline{z})}{i\left(1+|z|^2\right)} = \frac{\left(\zeta_1/\zeta_2-\overline{\zeta_1}/\overline{\zeta_2}\right)}{i\left(1+\frac{|\zeta_1|^2}{|\zeta_2|^2}\right)} = \frac{\left(\zeta_1/\zeta_2-\overline{\zeta_1}/\overline{\zeta_2}\right)\left(\zeta_2\,\overline{\zeta_2}\right)}{i\left(|\zeta_2|^2+|\zeta_1|^2\right)} = \frac{\zeta_1\,\overline{\zeta_2}\,\zeta_2\,\overline{\zeta_1}}{i\left(|\zeta_2|^2+|\zeta_1|^2\right)}$$

$$Z = \frac{|z|^2-1}{|z|^2+1} = \frac{|\zeta_1/\zeta_2|^2-1}{|\zeta_1/\zeta_2|^2+1} = \frac{\left(|\zeta_1/\zeta_2|^2-1\right)|\zeta_2|^2}{\left(|\zeta_1/\zeta_2|^2+1\right)|\zeta_2|^2} = \frac{|\zeta_1|^2-|\zeta_2|^2}{|\zeta_1|^2+|\zeta_2|^2}$$

P. 76 *You may readily check [exercise] that these formulas may be inverted to yield*

$$\begin{pmatrix} T+Z & X+iY \\ X-iY & T-Z \end{pmatrix} = 2\begin{pmatrix} \zeta_1\,\overline{\zeta_1} & \zeta_1\,\overline{\zeta_2} \\ \zeta_2\,\overline{\zeta_1} & \zeta_2\,\overline{\zeta_2} \end{pmatrix} = 2\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\begin{pmatrix} \overline{\zeta_1} & \overline{\zeta_2} \end{pmatrix}$$

I have to acknowledge that I was led through this exercise by Vasco (P.c. VDGAF Forum, August 2022). There may be differences, but I hope they are not significant.

The space time symbols T, Z, X, and Y may be thought of as measurements performed by an observer. The result in the exercise above can be rewritten as (6.17)

$$\begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} = 2\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}^*$$

where $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}^*$ is the conjugate transpose.

It helps to remember that the column entries of the conjugate transpose are actually conjugates of the source coordinates. In the stereographic map, the conjugate transpose of a matrix is equal to the matrix inverse, i.e., $[M]^* = [M]^{-1}$.

The matrix seems to appear out of nowhere, but it is easily recognized as the matrix whose determinant is $T^2 - \left(X^2+Y^2+Z^2\right) = I^2$. Note that the style of the brackets may have a special meaning. In Needham's words from *Visual Complex Analysis*, *We use (ROUND) brackets for a real matrix corresponding to a linear transformation of* $\mathbb{R}^2$ *or of* $\mathbb{C}$*, and we use [SQUARE] brackets for a (generally) complex matrix corresponding to a Möbius transformation of* $\mathbb{C}$ (*VCA*, p. 157). A Möbius transformation cannot be interpreted as a linear transformation of $\mathbb{R}^2$. That said, in this section of *VDGF*, he uses both round and square brackets for column vectors, square brackets for [M], and round brackets for the spacetime matrix with T, X, Y, and Z. Perhaps the square brackets used for the column vectors at the top of p. 77 were unintended, but I have reproduced them in (2). Elsewhere, I have followed Needham's usage.

It is given that (1)

$$T = |\zeta_1|^2 + |\zeta_2|^2 \qquad X = \zeta_1 \overline{\zeta_2} + \zeta_2 \overline{\zeta_1}$$

$$Y = i(\zeta_1 \overline{\zeta_2} + \zeta_2 \overline{\zeta_1}) \qquad Z = |\zeta_1|^2 - |\zeta_2|^2$$

and $\begin{bmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_1 \end{bmatrix} = [M]\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$. (2)

How does one invert the four formulas? Treating them as Möbius transformations did not appear to work. That is, I could not see how to rewrite them in the inversion form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

After consultation with Vasco, I realized that the intended inversion was one that would take T, X, Y, and Z back to expressions in z, but the z had to be written with the homogeneous (projective) coordinates. But this “inversion” is not entirely straightforward, because the full inversion to expressions in $z = \zeta_1/\zeta_2(\zeta_1$, corresponding to $(z) = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$, could only be accomplished for the matrix $\begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix}$ and it’s transform. It may or may not seem a bit misleading to speak of inversion of the four formulas.

The calculation of the inverse that takes the matrix with T, X, Y, and Z to a matrix in terms of $z = \zeta_1/\zeta_2$ using the formulas goes as follows:

$$\begin{pmatrix} T+Z & X+iY \\ X-iY & T-Z \end{pmatrix} = 2\begin{pmatrix} \zeta_1\overline{\zeta_1} & \zeta_1\overline{\zeta_2} \\ \zeta_2\overline{\zeta_1} & \zeta_2\overline{\zeta_2} \end{pmatrix} = 2\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}(\overline{\zeta_1}\ \ \overline{\zeta_2}) = 2\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}^*$$

We could write this as 2 (z) (z)*. Remember that $(z)^* = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}^* = (\overline{\zeta_1}, \overline{\zeta_2})$. Use tildes to write the matrix of the second observer

$$\begin{pmatrix} \tilde{T}+\tilde{Z} & \tilde{X}+i\tilde{Y} \\ \tilde{X}-i\tilde{Y} & \tilde{T}-\tilde{Z} \end{pmatrix} = 2\begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix}\begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix}^* \qquad (3)$$

and apply the substitution from (2).

$$2\,[M]\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\left\{\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}[M]\right\}^* = 2\,[M]\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}^*[M]^*$$

$$= [M]\begin{pmatrix}T+Z & X+iY\\ X-iY & T-Z\end{pmatrix}[M]^* = \begin{pmatrix}\tilde{T}+\tilde{Z} & \tilde{X}+i\,\tilde{Y}\\ \tilde{X}-i\,\tilde{Y} & \tilde{T}-\tilde{Z}\end{pmatrix}$$

Needham wrote “Because det[*M*] = 1 = det[*M*]*, it follows from (6.18) that this *linear transformation* preserves the spacetime interval” (p. 77). He appears to refer to (6.19) as the linear transformation, as he does in the sentence preceding it. It was not clear to me why

$$[M]\begin{pmatrix}T+Z & X+\mathrm{iY}\\ X-\mathrm{iY} & T-Z\end{pmatrix}[M]^*$$

should be regarded as a linear transformation. In VCA, p. 158 he wrote that a linear transformation of $\mathbb{C}^2$ is represented by a complex 2×2 matrix:

$$\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}\mapsto\begin{bmatrix}\mathrm{w}_1\\ \mathrm{w}_2\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}=\begin{bmatrix}\mathrm{a}\zeta_1 & \mathrm{b}\zeta_2\\ \mathrm{c}\zeta_1 & \mathrm{d}\zeta_2\end{bmatrix}$$

This linear transformation of $\mathbb{C}^2$ induces *a non-linear transformation of* $\mathbb{C}$, which is the most general Möbius transformation $\frac{\mathrm{a}z+b}{\mathrm{c}z+d}$. So we see that the linear transformations occur only in $\mathbb{C}^2$. Then it is easy to see that (3)

$$2\begin{pmatrix}\tilde{\zeta}_1\\ \tilde{\zeta}_2\end{pmatrix}\begin{pmatrix}\tilde{\zeta}_1\\ \tilde{\zeta}_2\end{pmatrix}^*$$

is a linear transformation in $\mathbb{C}^2$. And therefore, the substitution matrix

$$2\,[M]\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}^*[M]^*$$

is also a linear transformation in $\mathbb{C}^2$. It follows that

$$[M]\begin{pmatrix}T+Z & X+\mathrm{iY}\\ X-\mathrm{iY} & T-Z\end{pmatrix}[M]^*, \text{ as well as }\begin{pmatrix}\tilde{T}+\tilde{Z} & \tilde{X}+i\,\tilde{Y}\\ \tilde{X}-i\,\tilde{Y} & \tilde{T}-\tilde{Z}\end{pmatrix}$$

are linear transformations in $\mathbb{C}^2$, and therefore one can describe them as linear transformations, but not as linear transformations in $\mathbb{C}$! And the most general Möbius transformation is non-linear. At a more fundamental level, a transformation is linear if it does one or more of the following: (a) multiplies variables or constants by constants; (b) adds constants to variables or constants, or (c) adds constants to variables times constants. A matrix multiplication which does this and no more may be described as a linear transformation. The general Möbius transformation does not qualify. The reason may be seen in the decomposition of M found on p. 124 of VCA. Step (ii) is $z \mapsto (1/z)$. There is no constant by which one can multiply z in order to effect this transformation. In support of this point, Vasco pointed out that on p. 157 of *VDGF*, one finds "$M(z) = -(1/z)$, which is certainly not a linear transformation of $\mathbb{C}$". The negative sign has no bearing on the issue, as it is merely multiplication by the constant (-1).

The rest of p. 77 seems unproblematic.

P. 81. ¶ 2, [exercise] *or else we can use the intrinsic metric curvature formula (4.10) to confirm [exercise] that* $\mathcal{K} = -1$.

$$d\hat{s}^2 = \frac{\sin^2\phi\, d\theta^2 + d\phi^2}{\cos^2\phi}$$

Find A and B as the coefficients of dθ and dϕ.

$$d\hat{s}^2 = \frac{\sin^2\phi\, d\theta^2 + d\phi^2}{\cos^2\phi} = \frac{\sin^2\phi}{\cos^2\phi}\, d\theta^2 + \frac{1}{\cos^2\phi}\, d\phi^2$$

$$A = \tan\phi, \qquad B = \frac{1}{\cos\phi}$$

$$AB = \frac{\tan\phi}{\cos\phi}$$

$$\mathcal{K} = -\frac{\cos\phi}{\tan\phi}\left[\partial_\phi\left(\frac{\partial_\phi \tan\phi}{\frac{1}{\cos\phi}}\right) + \partial_\theta\left(\frac{\partial_\theta\left(\frac{1}{\cos\phi}\right)}{\tan\phi}\right)\right]$$

$$= -\frac{\cos\phi}{\tan\phi}\left[\partial_\phi\left(\frac{\sec^2\phi}{\sec\phi}\right) + \partial_\theta\left(\frac{0}{\tan\phi}\right)\right]$$

$$-\frac{\cos\phi}{\tan\phi}\left[\partial_\phi \sec\phi\right] = -\frac{\cos\phi}{\tan\phi}[\sec\phi\tan\phi]$$

$$= -\cos\phi\sec\phi = -1$$