Notes on Needham, VDGF, Chapter 5.
April 20, 2002

P. 52, bottom

$$\frac{-dX}{d\sigma} = \frac{X}{R} \quad \Longrightarrow \frac{dX}{X} = -\frac{d\sigma}{R}$$

Integrate both sides:

$\ln(X) = -\sigma / R + \alpha$, where α is a constant

When X = R, $\sigma = 0$, so $\alpha = \ln(R)$ and $e^{\alpha} = R$.

Raise both sides to power of e.

$$e^{\ln(x)} = e^{-\sigma/R+\alpha} = e^{\alpha}\, e^{-\sigma/R}$$

$$X = \mathrm{R}e^{-\sigma/R}$$

P. 53, (5.2)

Since the coordinates (x, σ) are orthogonal, apply (4.9), p. 37, substituting (x, σ) for (u, v) and substituting $A = X$ and $B = 1$.

$$d\hat{s}^2 = A^2\, du^2 + B^2\, dv^2 \qquad (4.9)$$

$$d\hat{s}^2 = X^2\, dx^2 + d\sigma^2$$

$$= \left(\mathrm{R}e^{-\sigma/R}\right)^2 dx^2 + d\sigma^2 \qquad (5.2)$$

$$\mathcal{K} = -\frac{1}{\mathrm{R}e^{-\sigma/R}\cdot 1}\left[\partial_\sigma\left(\frac{\partial_\sigma\left(\mathrm{R}e^{-\sigma/R}\right)}{1}\right) + \partial_x 0\right]$$

$$= -\frac{1}{\mathrm{R}e^{-\sigma/R}}\,\partial_\sigma\left[\mathrm{R}e^{-\sigma/R}(-1/R)\right] = -\frac{1}{\mathrm{R}e^{-\sigma/R}}\,\partial_\sigma\left[-e^{-\sigma/R}\right]$$

$$= -\frac{1}{\mathrm{R}e^{-\sigma/R}}\left(-e^{-\sigma/R}(-1/R)\right] = -\frac{1}{R^2} \qquad (5.3)$$

P. 62. *Confirm that this is indeed $\mathbb{H}^2$ by using the conformal curvature formula (4.16) to verify [exercise] that this surface has constant negative curvature $\mathcal{K} = -1$.*

The equation $d\hat{s} = \frac{2}{1-r^2}$ ds implies, from p. 41, ¶ 2, that $\Lambda = \frac{2}{1-r^2}$. Then, from (4.16), and bearing in mind that $r = 1$,

$$\mathcal{K} = -\frac{\nabla^2 \ln\Lambda}{\Lambda^2} = -\frac{\nabla\left[2\,r/\left(1-r^2\right)\right]}{\Lambda^2} = -\frac{2\times\left(1+r^2\right)/\left(r^2-1\right)^2}{4/\left(1-r^2\right)^2} = -\frac{\left(1+r^2\right)}{2} = -1$$