Needham, Chapter 4, Notes.
G. Palmer, October 7, 2022, 5:30 pm PDT

P. 33. ¶ 2. *We can now use this fact to derive a metric of the sphere in terms of polar coordinates* $(r,\ \theta)$ *in the projective map. It is not hard to accomplish this by calculation [exercise], ...*

Previously in the paragraph, we learned that the central projections of lines of latitude and longitude from the sphere to the complex tangent plane are orthogonal. The result on the plane is a family of concentric circles centred on the origin and intersected at right angles by a family of rays from the origin. Each circle has a distance r from the origin and each ray has an angle θ from the x axis. The orthogonal sides of the black triangle in [4.2] are plotted by moving z along short segments of a circle and then a ray, and then plotting the hypotenuse.

By inspection of 4.1, we have $s_{\text{lat}} = \mathrm{r}\theta$ on the plane, so we can write an equation for an infinitesimal arc ds_1 on the plane corresponding to an infinitesimal line of latitude $d\,\hat{s}_1$ on the sphere.

$$\mathrm{ds}_1 = r\,\mathrm{d}\theta$$

$$d\,\hat{s}_1 = \frac{R}{H}\,\mathrm{ds}_1 = \frac{R}{\sqrt{R^2+r^2}}\,r\,\mathrm{d}\theta$$

A similar equation for an arc emanating from z corresponds to a line of longitude of the sphere.

$$s_{\text{lon}} = \mathrm{R}\phi$$

Since $\phi = \tan^{-1}(r/R)$,

$$\mathrm{d}\phi = \frac{1}{\left(1+\frac{r^2}{R^2}\right)R}\,\mathrm{dr}$$

$$d\,\hat{s}_2 = R\,\mathrm{d}\phi = \frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,\mathrm{dr}$$

Then the metric on the surface is

$$(d\,\hat{s}_2)^2 = \left(\frac{R}{\sqrt{R^2+r^2}}\,\mathrm{r}\mathrm{d}\theta\right)^2 + \left(\frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,\mathrm{dr}\right)^2$$

$$= \frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,r^2\,\mathrm{d}\theta^2 + \frac{1}{\left(1+\frac{r^2}{R^2}\right)^2}\,\mathrm{dr}^2$$

$$= \frac{1}{\left(1+\frac{r^2}{R^2}\right)}\left[\frac{1}{\left(1+\frac{r^2}{R^2}\right)}\,\mathrm{dr}^2 + r^2\,\mathrm{d}\theta^2\right]$$

This is (4.2)

P. 33 Derivation of (4.2)

It is given that $\frac{\delta\,\hat{s}_1}{\mathrm{r}\delta\theta} = \frac{R}{H}$ and $\frac{d\,\hat{s}_2}{\mathrm{dr}} = \frac{R^2}{H^2}$.

Then, $\delta\,\hat{s}_1 = \frac{\mathrm{Rr}\delta\theta}{H} = \frac{\mathrm{Rr}\delta\theta}{\sqrt{R^2+r^2}}$

and $d\,\hat{s}_2 = \frac{R^2\,\mathrm{dr}}{H^2} = \frac{R^2\,\mathrm{dr}}{R^2+r^2}$.

Use the infinitesimal notation for $\delta\,\hat{s}_1$ and $\delta\theta$.

$$d\,\hat{s}^2 = (d\,\hat{s}_1)^2 + (d\,\hat{s}_2)^2$$

$$= \left(\frac{\mathrm{Rrd}\theta}{\sqrt{R^2+r^2}}\right)^2 + \left(\frac{R^2\,\mathrm{dr}}{R^2+r^2}\right)^2$$

$$= \frac{R^2\,r^2\,\mathrm{d}\theta^2}{R^2+r^2} + \frac{R^2}{R^2+r^2}\cdot\frac{R^2\,\mathrm{dr}^2}{R^2+r^2}$$

$$= \frac{R^2}{R^2+r^2}\left(r^2\,\mathrm{d}\theta^2 + \frac{R^2\,\mathrm{dr}^2}{R^2+r^2}\right)$$

$$= \frac{1}{1+\left(\frac{r}{R}\right)^2}\left(\frac{\mathrm{dr}^2}{1+\left(\frac{r}{R}\right)^2} + r^2\,\mathrm{d}\theta^2\right)$$

P. 33 *Returning to [4.1], we can now quantify the amount of distortion induced by the map, by looking at the shape of the small ellipse* $\in$. *As C shrinks, you should now be able to confirm [exercise] that*

$$\left(\frac{\textit{major axis of} \in}{\textit{minor axis of} \in}\right) \asymp sec\phi$$

so as C moves north towards the rim $\left(\phi = \frac{\pi}{2}\right)$, *the distortion of shapes occurs without limit.*

By inspection of 4.2, $\sec\phi = 1/\cos\phi = H/R$. Show that as the radius of C shrinks,

$$\left(\frac{\textit{major axis of} \in}{\textit{minor axis of} \in}\right) \asymp H/R$$

Let $\delta\phi$ be the angle from c (center at equator, as in [4.2]) to any point on the circumference of the small circle C on the hemisphere $\mathcal{S}$. Since $r_C = R\delta\phi$, as $\delta\phi$ shrinks to 0, so does r_C.

Let r_{min} and r_{maj} be the minor and major axes of the ellipse $\in$. Then, as $\delta\phi$ shrinks to 0,

$$r_{\text{min}} \asymp H\,\delta\phi$$

From p. 33,

$$\frac{\varepsilon}{\delta \text{r}} \asymp \frac{R}{H}$$

From [4.2], one can see that in this case that $\varepsilon \asymp H\,\delta\phi$.

Then

$$r_{\text{maj}} = \delta\text{r} \asymp \frac{\varepsilon \text{H}}{R} = \frac{(H\,\delta\phi)\,H}{R} = \frac{H^2\,\delta\phi}{R}$$

$$\frac{r_{\text{maj}}}{r_{\text{min}}} = \frac{\left(\frac{H^2\,\delta\phi}{R}\right)}{H\,\delta\phi} = \frac{H}{R} = \sec(\phi)$$

QED

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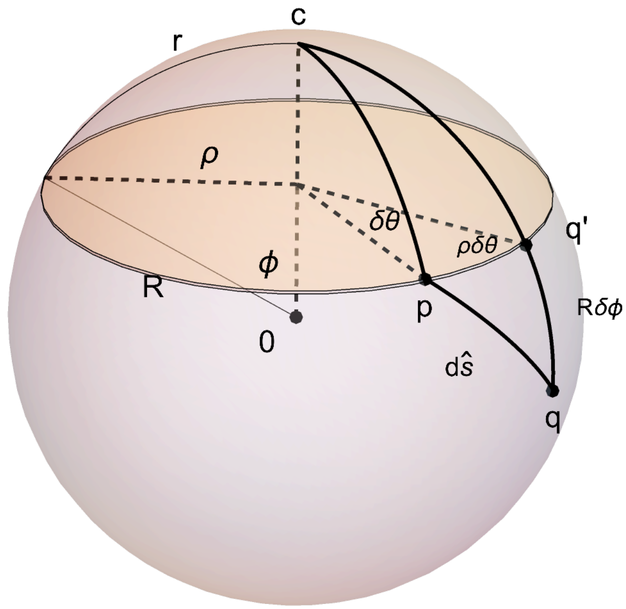

Figure 1. Distance dŝ from p to q on sphere

P. 34 *With the assistance of [2.4], you should now easily find [exercise] that the metric formula telling us the true distance on the sphere corresponding to neighboring points in the map is*

$$d\hat{s}^2 = R^2\left[\sin^2\phi d\theta^2 + d\phi^2\right].$$

See Figure 1.

$$\rho = \text{R} \sin\phi,$$

$d\hat{s}^2$ is the infinitesimal segment of a great circle from p to q.

The other two sides of the infinitesimal spherical triangle pqq' are $d\hat{s}_1$ and $d\hat{s}_2$.

$$d\hat{s}_1 = \rho \text{d}\theta \asymp \rho\delta\theta$$

$$d\hat{s}_2 = \text{Rd}\phi \asymp \text{R}\delta\phi$$

Then, as the spherical triangle pqq' shrinks

$$d\hat{s}^2 = d\hat{s}_1{}^2 + d\hat{s}_2{}^2 \asymp (R \sin\phi\ \delta\theta)^2 + (R\delta\phi)^2$$

which can be rewritten with infinitesimals as

$$d\hat{s}^2 = R^2\left[\sin^2\phi\ d\theta^2 + d\phi^2\right]$$

QED

P. 38, **¶ 5** *Although the calculation is longer, we encourage you to try your own hand at applying this formula* [(4.10)] *to the projective metric formula (4.2) for the sphere; this too should yield* $\mathcal{K} = +\left(1/R^2\right)$.

Let u = θ and let v = r.

From p. 33,

$$\frac{d\hat{s}_1}{\mathrm{r}d\theta} \asymp \frac{R}{H} \Longrightarrow \frac{d\hat{s}_1}{\mathrm{d}\theta} \asymp \frac{\mathrm{rR}}{H} \Longrightarrow \frac{d\hat{s}_1}{\mathrm{du}} \asymp \frac{\mathrm{rR}}{H} = A$$

$$\frac{d\hat{s}_2}{\mathrm{dr}} \asymp \frac{R^2}{H^2} \Longrightarrow \frac{d\hat{s}_2}{\mathrm{dv}} \asymp \frac{R^2}{H^2} = B$$

Substitute for A, B, u, and v into (4.10)

$$\mathcal{K} = -\frac{H^3}{\mathrm{rR}^3}\left(\partial_r\left[\partial_r\left(\frac{\mathrm{rR}}{H}\right)\Big/\frac{R^2}{H^2}\right]+\left(\partial_\theta\left[\partial_\theta\left(\frac{R^2}{H^2}\right)/\frac{\mathrm{rR}}{H}\right]\right)\right.$$

Because $H^2 = r^2 + R^2$,

$$\mathcal{K} = -\frac{\left(r^2+R^2\right)^{3/2}}{\mathrm{rR}^3}\left(\partial_r\left[\frac{R^3}{\left(r^2+R^2\right)^{3/2}}\cdot\frac{r^2+R^2}{R^2}\right] + \partial_\theta[0]\right)$$

$$= -\frac{\left(r^2+R^2\right)^{3/2}}{\mathrm{rR}^3}\left[-\frac{r\,R}{\left(r^2+R^2\right)^{3/2}} + 0\right] = \frac{1}{R^2}$$

QED

The partials were calculated with Mathematica, though it is easy to see that the second order partial wrt u is zero, as r and R do not vary with u or θ.

Calculations with Mathematica

$\partial_r A$: $\dfrac{R^3}{\left(r^2 + R^2\right)^{3/2}}$

$\partial_r[(\partial_r A)/B]$: $-\dfrac{r\,R}{\left(r^2 + R^2\right)^{3/2}}$

$\partial_\theta B$: 0

$\mathcal{K}$: $\dfrac{1}{R^2}$

P. 38 *But how shall we convert areas? Well, in [4.3] a small rectangle in the map has an area $\delta u \delta v$, and on the surface the corresponding parallelogram has an area that is ultimately equal to $(\mathcal{A}\delta u)\,(B\delta v \sin\omega)$. Thus, using (4.7) the formula for the infinitesimal element of the area $d\mathcal{A}$ on the surface is*

$$d\mathcal{A} = \sqrt{(AB)^2 - F^2}\; du dv.$$

This puzzled me. I wondered whether $d\,\hat{s}^2 \asymp d\mathcal{A}$, as both appear to have the unit length^2, and I could not see how to make substitutions from (4.7) into (4.11) or *vice versa*. It turned out to be quite simple. $\delta\,\hat{s}^2$ in [4.3] (corresponding to $d\,\hat{s}^2$ in (4.7) is just an abstract area, not shown, resulting from the square of the distance plotted as the long side of the black triangle in [4.3]. It is a conventional way of writing the metric. $d\mathcal{A}$ is the area of the parallelogram created on the black triangle. The white triangle is there to establish the length of the height of the black triangle as $(B\delta v \sin\omega)$ where $\delta\,\hat{s}_1$ is the base. Then one can use the old formula (1/2) bh on the black triangle and double it to obtain $(\mathcal{A}\delta u)\,(B\delta v \sin\omega)$.

$$d\mathcal{A} \asymp (A\delta u)\,(B\delta v \sin\omega) = \sqrt{(AB)^2 \sin^2\omega}\;\delta u \delta v$$

$$= \sqrt{(AB)^2\left(1 - \cos^2\omega\right)}\;\delta u \delta v$$

$$= \sqrt{(AB)^2 - (AB)^2\cos^2\omega}\;\delta u \delta v$$

Substitute F for ABcos ω, from (4.7), and let $\delta u = du$ and $\delta v = dv$.

$$= \sqrt{(AB)^2 - F^2}\; du dv$$

Done.

P. 41. *Since now $A = B = \Lambda$, we easily find [exercise] that (4.10) simplifies to*

$$\mathcal{K} = -\frac{\nabla^2 \ln\Lambda}{\Lambda^2} \qquad (4.16)$$

From p. 38,

$$\mathcal{K} = -\frac{1}{AB^2}\left(\partial_v\left(\frac{\partial_v A}{B}\right) + \partial_u\left(\frac{\partial_u B}{A}\right)\right) \qquad (4.10)$$

Since $A = B = \Lambda$, one can replace the A's and B's of (4.10) with Λ.

$$\mathcal{K} = -\frac{1}{\Lambda^2}\left(\partial_v\left(\frac{\partial_v \Lambda}{\Lambda}\right) + \partial_u\left(\frac{\partial_u \Lambda}{\Lambda}\right)\right)$$

$$= -\frac{1}{\Lambda^2}\left(\partial_v\left(\partial_v \Lambda \cdot \frac{1}{\Lambda}\right) + \partial_u\left(\partial_u \Lambda \cdot \frac{1}{\Lambda}\right)\right)$$

$$= -\frac{1}{\Lambda^2}\left(\partial_v\left(\partial_v \ln\Lambda\right) + \partial_u\left(\partial_u \ln\Lambda\right)\right)$$

$$= -\frac{1}{\Lambda^2}\left(\partial_v^2 \ln\Lambda + \partial_u^2 \ln\Lambda\right)$$

$$= -\frac{\nabla^2 \ln\Lambda}{\Lambda^2} \quad \text{QED}$$

P. 43. To my mind, Figure 2 provides a clearer illustration of the local effects of amplitwist. In the case of z^2, the amplitwist is $(z^2)' = 2\,z = 2\,re^{i\theta}$. An infinitesimal vector dz anchored at z can be multiplied by the amplitwist to obtain its image at z^2. The half circle is on the unit circle.

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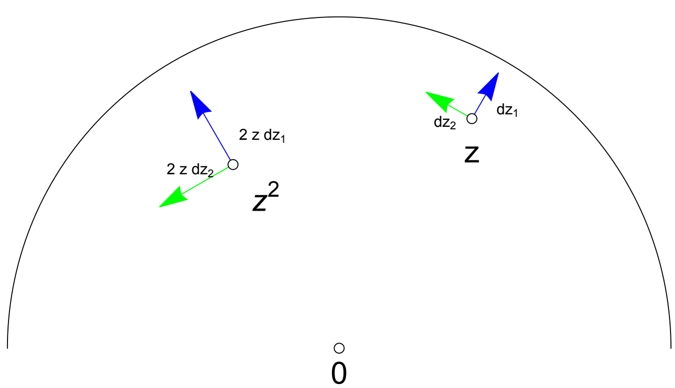

Figure 2. Local effect of amplitwist
on small movements of z

Now compare this to a more algebraic approach. Suppose we add dz to z prior to squaring. Then

$$(z + dz)^2 = z^2 + 2\,z\,dz + dz^2$$

If we think of each term of the sum as a vector, then there are now two small vectors to consider: 2 z dz anchored at z^2 and dz^2 anchored at $z^2 + 2z\,dz$. Figure 3 shows that dz^2 amplifies the sum by a tiny amount. Hence, its omission from amplitwist introduces a tiny error into the calculation of the local effect of z^2. As dz is allowed to go to zero, one can, ultimately, disregard it as insignificant compared to the derivative 2 z dz.

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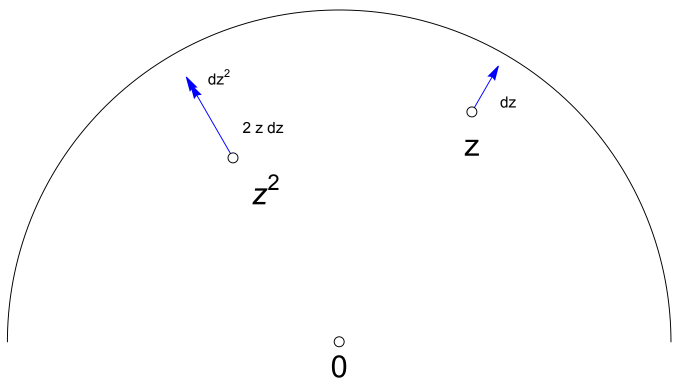

Figure 3. Local effect of z^2
on small movements of z

P. 44. *It is straightforward [exercise] to generalize this geometric argument to deduce that* $(z^m)' = mz^{m-1}$.

$\tau = \text{twist} = m\theta - \theta = (m-1)\,\theta$

Let A = amplification, T = twist, and AT = amplitwist.

Since angles are multiplied by m, the image arc will subtend angle $m\delta\theta$, and since it now lies on a circle of radius r^n, the length of this image edge is ultimately $r^n(m\delta\theta)$. The quotient is

$r^m(m\delta\theta)/(r\delta\theta) = mr^{m-1}$

$\left(\text{image edge} \asymp mr^{m-1}(\text{original edge}) \Longrightarrow \alpha = A = mr^{m-1}\right)$

$(z^m)' = \text{AT of } z^m = Ae^{iT} = mr^{m-1}\,e^{i(m-1)\,\theta} = mz^{m-1}$

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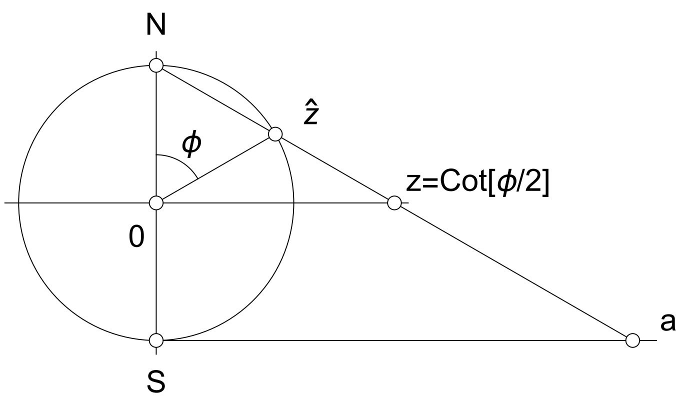

Figure 3. Stereographic projection on cross section of Riemann sphere

P 44. footnote 11. *This [exercise] alters the map only by a constant scale factor of 2.*

By inspection of figure 2, the right triangles aNS and zN0 are similar. The height of aNS is twice that of zN0. Therefore, the other dimensions of the triangle aNS are also scaled by 2. This is true at all θ-angles. Therefore, the map on the tangent plane at S is altered by a constant scale of 2. For the calculation of z from $\hat{z}$, see (21), *Visual Complex Analysis*, p. 147.

P. 47, ¶2. *Confirm [$\mathcal{K} = (1/R^2)$...in two ways. First write $r^2 = x^2 + y^2$ and use our original Cartesian form of the Laplacian operator, (4.15).*

$$\nabla^2 = \partial_u^2 + \partial_v^2 \qquad (4.15)$$

$$\mathcal{K} = -\frac{\nabla^2 \ln \Lambda}{\Lambda^2} \qquad (4.16)$$

From (4.22) let $\Lambda = \frac{2}{1+(r/R)^2}$.

$$\partial_x \Lambda = -\frac{\Lambda^2 x}{R^2}, \qquad \partial_y \Lambda = -\frac{\Lambda^2 y}{R^2}$$

$$\partial_x \ln\Lambda = \frac{1}{\Lambda}\,\partial_x \Lambda = -\frac{\Lambda x}{R^2}, \qquad \partial_y \ln\Lambda = -\frac{\Lambda y}{R^2}$$

$$\partial_x^2 \ln\Lambda = \partial_x\left(-\frac{\Lambda x}{R^2}\right) = -\frac{1}{R^2}[(\partial_x \Lambda)\,x + \Lambda]$$

$$= -\frac{1}{R^2}\left[\left(-\frac{\Lambda^2 x}{R^2}\right)x + \Lambda\right] = -\frac{1}{R^2}\left[-\frac{\Lambda^2 x^2}{R^2} + \Lambda\right]$$

$$\partial_y^2 \ln\Lambda = -\frac{1}{R^2}\left[-\frac{\Lambda^2 y^2}{R^2} + \Lambda\right]$$

$$\nabla^2 \ln \Lambda = \partial_x^2 \ln\Lambda + \partial_y^2 \ln\Lambda = -\frac{1}{R^2}\left[-\frac{\Lambda^2 x^2}{R^2} + \Lambda - \frac{\Lambda^2 y^2}{R^2} + \Lambda\right]$$

$$= -\frac{1}{R^2}\left[-\frac{\Lambda^2 r^2}{R^2} + 2\Lambda\right]$$

$$\mathcal{K} = -\frac{\nabla^2 \ln\Lambda}{\Lambda^2} = -\left(-\frac{1}{R^2}\left[-\Lambda^2\left(\frac{r}{R}\right)^2 + 2\Lambda\right]\Big/\Lambda^2\right) = \frac{1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + \frac{2}{\Lambda}\right]$$

$$= \frac{1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + 2\times\left(1+\left(\frac{r}{R}\right)^2\right)/2\right] = \frac{1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + \left(1+\left(\frac{r}{R}\right)^2\right)\right]$$

$$= \frac{1}{R^2} \qquad \text{QED}$$

Second, use the fact that in polar coordinates the Laplacian instead takes the form

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\partial_\theta^2$$

First calculate the partials with respect to r using the conformal metric supplied in (4.22).

$$\partial_r \Lambda = -r\Lambda^2/R^2$$

$$\partial_r \ln\Lambda = \frac{1}{\Lambda}\,\partial_r\Lambda = \frac{1}{\Lambda}\left(-r\Lambda^2/R^2\right) = -\Lambda r/R^2$$

$$\partial_r^2 \ln\Lambda = \partial_r\left(-\Lambda r/R^2\right) = -\frac{1}{R^2}\left[(\partial_r\Lambda)\, r + \Lambda\right]$$

Ignore the third term because the scaling component Λ of the metric for stereographic projections has no θ component, and substitute the partials into the Laplacian second order differential operator in the numerator of $\mathcal{K}$ for the conformal mapping.

$$\mathcal{K} = -\frac{\nabla^2\ln\Lambda}{\Lambda^2} = -\frac{\partial_r^2\ln\Lambda + \frac{1}{r}\,\partial_r\ln\Lambda}{\Lambda^2} = -\frac{-\frac{1}{R^2}\left[(\partial_r\Lambda)\, r + \Lambda\right] + \frac{1}{r}\left(-\Lambda r/R^2\right)}{\Lambda^2}$$

$$= -\frac{-\frac{1}{R^2}\left[\left(-r\Lambda^2/R^2\right) r + \Lambda\right] + \left(-\Lambda/R^2\right)}{\Lambda^2} = -\frac{r^2\,\Lambda^2/R^2 - \Lambda - \Lambda}{R^2\,\Lambda^2}$$

$$= -\frac{1}{R^2}\left[\frac{r^2}{R^2} - \frac{2}{\Lambda}\right] = -\frac{1}{R^2}\left[\left(\frac{r}{R}\right)^2 - \frac{2\times\left(1+(r/R)^2\right)}{2}\right]$$

$$= -\frac{1}{R^2}\left[\left(\frac{r}{R}\right)^2 - \left(1+\left(\frac{r}{R}\right)^2\right)\right] = \frac{1}{R^2}$$

QED

P. 48, ¶ 1. [exercise]

The triangles with hypotenuses Nz and Nz are similar right triangles. Therefore, the ratios of the lengths of corresponding sides are equal for all three sides. For example, the ratio of the bases equal the ratio of the heights, i.e.

$$\frac{|z|}{|z'|} = \frac{1}{1-Z}$$

Then, because z and z' have the same direction,

$$\frac{|z|e^{i\theta}}{|z'|e^{i\theta}} = \frac{1}{1-Z}$$

but $|z|e^{i\theta} = x + iy$, and $|z'|e^{i\theta} = X + iY$, so

$$x + iy = \frac{X+iY}{1-Z}$$

P. 48, ¶ 2. [exercise]

Confirm that $|z|^2 = \frac{1+Z}{1-Z}$

From [4.12], use the Pythagorean theorem to write

$$|z'|^2 = 1 - Z^2 = (1-Z)\times(1+Z)$$

Since $\frac{|z|}{|z'|} = \frac{1}{1-Z}$

$$|z|^2 = \frac{|z'|^2}{(1-Z)^2} = \frac{(1-Z)\times(1+Z)}{(1-Z)^2} = \frac{(1+Z)}{(1-Z)}$$

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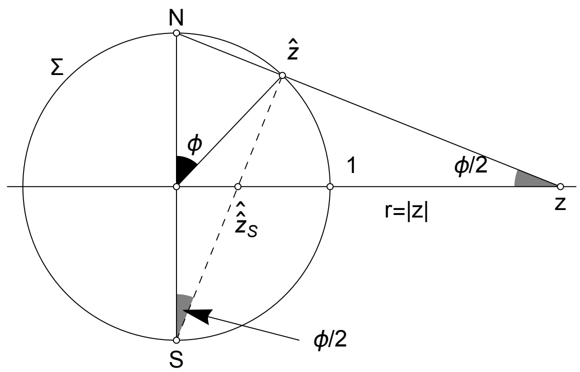

Figure 3. Cf. [23], p. 147

P. 49. *From [4.12b] it is clear [exercise] that the triangles $N\hat{z}S$ and $N0z$ are similar.*

The triangles share the angle at N. They have equal angles at S and z. They both have right triangles by construction. Hence, they are similar.

...because the angle $\angle NS\hat{z} = (\phi/2)$, it follows [exercise] that $r = \cot(\phi/2)$.

$$\sin(\phi/2) = \frac{1}{|N-z|}$$
$$\cos(\phi/2) = \frac{z}{|N-z|}$$
$$\cot(\phi/2) = \frac{z}{|N-z|} \cdot \left| N - z \right| = z$$

P. 50.* ¶ *2. *Substituting (4.25) into the equation of this plane we find* ***[exercise]*** *that the circle of intersection on Σ stereographically projects to a curve in ℂ whose equation is*

$$l2x + m2y + n(x^2 + y^2 - 1) = k(1 + x^2 + y^2)$$

Let $A = 1 + x^2 + y^2$

$l X + m Y + n Z = k$

$l \frac{2x}{A} + m \frac{2y}{A} + n \frac{x^2+y^2-1}{A} = k$ (substituting from 4.25 for X, Y, and Z)

$\mathrm{l2x} + \mathrm{m2y} + n(x^2 + y^2 - 1) = \mathrm{kA} = k(1 + x^2 + y^2)$

If k = n*, then* ***[exercise]*** *the circle on Σ passes through N and its image is a line (as it should be) with equation $lx + my = n$. If $k \neq n$, then we may complete the squares [exercise] to rewrite the equation as*

$$\left[x - \frac{l}{k-n}\right]^2 + \left[y - \frac{m}{k-n}\right]^2 = \frac{l^2+m^2+n^2-k^2}{|k-n|}$$

$$\text{l2x} + \text{m2y} + n\left(x^2 + y^2 - 1\right) = n\left(1 + x^2 + y^2\right)$$

$$\text{l2x+m2y} = \text{n}[1 + x^2 + y^2 - \left(x^2 + y^2 - 1\right)] = 2\,n$$

$$\text{lx} + \text{my} = n$$

Since the image is a line in ℂ, then the circle in Σ must pass through N as shown in [4.8] and set out in (4.21).

If k ≠ n,

$$\text{l2x} + \text{m2y} + n\left(x^2 + y^2 - 1\right) = k\left(1 + x^2 + y^2\right)$$

Expand and rearrange.

$$\left(\text{nx}^2 - \text{kx}^2 + \text{l2x}\right) + \left(\text{ny}^2 - \text{ky}^2 + \text{m2y}\right) = n + k$$

$$\left[(n-k)\,x^2 + \text{l2x}\right] + \left[(n-k)\,y^2 + \text{m2y}\right] = n + k$$

$$\left[(k-n)\,x^2 - \text{l2x}\right] + \left[(k-n)\,y^2 - \text{m2y}\right] = -(n + k) \quad \text{(times } -1\text{)}$$

$$\left[x^2 - \frac{\text{l2x}}{(k-n)}\right] + \left[y^2 - \frac{\text{m2y}}{(k-n)}\right] = \frac{-(n+k)}{(k-n)}$$

Complete the squares.

$$\left[x - \frac{l}{(k-n)}\right]^2 + \left[y^2 - \frac{m}{(k-n)}\right]^2 = \frac{-(n+k)\,(k-n)}{(k-n)^2} + \frac{l^2}{(k-n)^2} + \frac{m^2}{(k-n)^2}$$

$$= \frac{-(n+k)\,(k-n)+l^2+m^2}{(k-n)^2} = \frac{-\text{nk}+n^2-k^2+\text{nk}+l^2+m^2}{(k-n)^2}$$

$$= -\frac{n^2+l^2+m^2-k^2}{(k-n)^2}$$

This is what we wanted: $\left[x - \frac{l}{k-n}\right]^2 + \left[y - \frac{m}{k-n}\right]^2 = \frac{l^2+m^2+n^2-k^2}{(k-n)^2}$