Needham, VDGF, Chapter 39, Exercise 9, p. 467.
G. Palmer, July 24, 2026, 6 pm, PT

9. Area formula for surface f = const. *Let $\mathcal{S}$ be the surface with equation $f(x, y, z) = \text{const.}$*

(i) Show that the unit normal n to $\mathcal{S}$ is

$$\boldsymbol{n} = \frac{1}{\sqrt{(\partial_x f)^2 + (\partial_y f)^2 + (\partial_z f)^2}} \begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{pmatrix}$$

A gradient such as the vector in the equation is normal to the surface. The unit normal of the vector is the square root of the sum of the squared components. Therefore, the unit normal of $f(x, y, z) = \text{const.}$ is as written in the exercise.

*(ii) By thinking of **n** as the velocity of fluid flowing at unit speed orthogonally across $\mathcal{S}$, deduce that the area 2-form is*

$$\mathcal{A} = \frac{\partial_x f\, \boldsymbol{d}y \wedge \boldsymbol{d}z + \partial_y f\, \boldsymbol{d}z \wedge \boldsymbol{d}x + \partial_z f\, \boldsymbol{d}x \wedge \boldsymbol{d}y}{\sqrt{(\partial_x f)^2 + (\partial_y f)^2 + (\partial_z f)^2}}.$$

By (34.9) the area 2-form in three dimensions is:

$$\boldsymbol{\Psi} = \Psi_{23}(\boldsymbol{d}y \wedge \boldsymbol{d}z) + \Psi_{31}(\boldsymbol{d}z \wedge \boldsymbol{d}x) + \Psi_{12}(\boldsymbol{d}x \wedge \boldsymbol{d}y)$$

In (34.10) the components are written with a single raised index. In this instance, keep the x, y, z designations of the coordinate 1-forms.

$$\boldsymbol{\Psi} = \Psi^1(\boldsymbol{d}y \wedge \boldsymbol{d}z) + \Psi^2(\boldsymbol{d}z \wedge \boldsymbol{d}x) + \Psi^3(\boldsymbol{d}x \wedge \boldsymbol{d}y)$$

So that $\underline{\boldsymbol{\Psi}} = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}$.

On p. 378, $\underline{\boldsymbol{\Psi}}$ is defined as the velocity of the uniform flow of fluid through space and depicted as flowing in the direction of the z axis (orthogonally across $\mathcal{S}$). Then,

let $R = \sqrt{(\partial_x f)^2 + (\partial_y f)^2 + (\partial_z f)^2}$ and

let $\Psi^1 = \frac{\partial_x f}{R}$, $\Psi^2 = \frac{\partial_y f}{R}$, and $\Psi^3 = \frac{\partial_y f}{R}$, so that $\boldsymbol{n} = \frac{1}{R}\begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{pmatrix}$ and

$$A = n^1(\boldsymbol{d}y \wedge \boldsymbol{d}z) + n^2(\boldsymbol{d}z \wedge \boldsymbol{d}x) + n^3(\boldsymbol{d}x \wedge \boldsymbol{d}y),$$

and as was to be shown

$$A = \frac{\partial_x f\, \boldsymbol{d}y \wedge \boldsymbol{d}z + \partial_y f\, \boldsymbol{d}z \wedge \boldsymbol{d}x + \partial_z f\, \boldsymbol{d}x \wedge \boldsymbol{d}y}{\sqrt{(\partial_x f)^2 + (\partial_y f)^2 + (\partial_z f)^2}}.$$