Needham, VDGF, Chapter 39, Exercise 8, p. 466.
G. Palmer, July 10, 2026, 9:40 pm, PT

8. Need not factorize in $\mathbb{R}^{4}$. *In* $\mathbb{R}^4$*, let* $\mathbf{\Psi}$ *be a general 2-form. Show that* $\mathbf{\Psi}$ <u>cannot</u> *always be factorized as the wedge product of two 1-forms:* $\mathbf{\Psi} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.

(i) Show that $\mathbf{\Psi}$ *cannot always be factorized as the wedge product of two 1-forms. Hint: If it can be factorized, consider* $\mathbf{\Psi} \wedge \mathbf{\Psi}$*. But now look back at §35.7.*

If $\mathbf{\Psi}$ is factorizable it must be the product of two 1-forms. If it can be factorized (why not "factored") then in 4 dimensions it would have the general format as follows:

Let $\boldsymbol{d}t$, $\boldsymbol{d}x$, $\boldsymbol{d}y$, and $\boldsymbol{d}z$, be the basis 1-forms in four dimensions. Then the wedge product of two general 1-forms is a 2-form that sums 6 vector dimensions (basis wedge products) by the formula (1/2)n(n-1) where $n = 4$ (p. 377, ¶1).

Consider two 1-forms defined on four dimensions in $\mathbb{R}^4$:

$$\boldsymbol{\alpha} = a_0\, \boldsymbol{d}t + \alpha_1\, \boldsymbol{d}x + \alpha_2\, \boldsymbol{d}y + \alpha_3\, \boldsymbol{d}z$$
$$\boldsymbol{\beta} = \beta_0\, \boldsymbol{d}t + \beta_1\, \boldsymbol{d}x + \beta_2\, \boldsymbol{d}y + \beta_3\, \boldsymbol{d}z$$

Then, from pages 376 and 380,

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (\alpha_0\, \beta_1 - \alpha_1\, \beta_0)\, \boldsymbol{d}t \wedge \boldsymbol{d}x + (\alpha_0\, \beta_2 - \alpha_2\, \beta_0)\, \boldsymbol{d}t \wedge \boldsymbol{d}y + (\alpha_0\, \beta_3 - \alpha_3\, \beta_0)\, \boldsymbol{d}t \wedge \boldsymbol{d}z + (\alpha_1\, \beta_2 - \alpha_2\, \beta_1)\, \boldsymbol{d}x \wedge \boldsymbol{d}y + (\alpha_1\, \beta_3 - \alpha_3\, \beta_1)\, \boldsymbol{d}x \wedge \boldsymbol{d}z + (\alpha_2\, \beta_3 - \alpha_3\, \beta_2)\, \boldsymbol{d}y \wedge \boldsymbol{d}z$$

This is obviously factorable, because it originated as the wedge product of two 1-forms. But if the corresponding coefficients of the basis wedge products differ from $\left(a_i\, b_j - a_j\, b_i\right)$, then the 2-form is not factorable into two 1-forms.

(ii) If $\mathbf{\Psi} \wedge \mathbf{\Psi} = 0$*, prove that* $\mathbf{\Psi}$ *can be factorized as the wedge product of two 1-forms.*

If $\mathbf{\Psi} \wedge \mathbf{\Psi} = 0$, then let $\mathbf{\Psi} = \boldsymbol{\varphi} \wedge \boldsymbol{\gamma}$ and write

$\mathbf{\Psi} \wedge \mathbf{\Psi} = (\boldsymbol{\varphi} \wedge \boldsymbol{\gamma}) \wedge (\boldsymbol{\varphi} \wedge \boldsymbol{\gamma}) = \boldsymbol{\varphi} \wedge \boldsymbol{\gamma} \wedge \boldsymbol{\varphi} \wedge \boldsymbol{\gamma} = -\boldsymbol{\varphi} \wedge \boldsymbol{\varphi} \wedge \boldsymbol{\gamma} \wedge \boldsymbol{\gamma} = -0 \wedge 0 = 0$*, by p. 373, 2nd box, and by (35.4).*

This shows that if $\mathbf{\Psi} \wedge \mathbf{\Psi} = 0$, $\mathbf{\Psi}$ can be factorized as the wedge product of two 1-forms.

(iii) Show that $\mathbf{\Psi}$ can always be expressed as the sum of two wedge products:

$$\boldsymbol{\Psi} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + \boldsymbol{\gamma} \wedge \boldsymbol{\delta}$$

If $\boldsymbol{\Psi}$ is a 2-form in 4 dimensions, one can only write three unique equations:

(1) $\boldsymbol{\Psi} = \alpha \boldsymbol{d}t \wedge \beta \boldsymbol{d}x + \gamma \boldsymbol{d}y \wedge \delta \boldsymbol{d}\delta = \alpha\beta \boldsymbol{d}t \wedge \boldsymbol{d}x + \gamma\delta \boldsymbol{d}y \wedge \boldsymbol{d}\delta$

$= \Psi_1\, \boldsymbol{d}t \wedge \boldsymbol{d}x + \Psi_2\, \boldsymbol{d}y \wedge \boldsymbol{d}\delta$

or

(2) $\boldsymbol{\Psi} = \Psi_1\, \boldsymbol{d}t \wedge \boldsymbol{d}y + \Psi_2\, \boldsymbol{d}x \wedge \boldsymbol{d}\delta$

or

(3) $\boldsymbol{\Psi} = \Psi_1\, \boldsymbol{d}t \wedge \boldsymbol{d}\delta + \Psi_2\, \boldsymbol{d}x \wedge \boldsymbol{d}y$

where Ψ_1 and Ψ_2 may be different for each of the three equations. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ each have one and only one of the four available 1-form bases. Since $\boldsymbol{\Psi}$ is a two form, no wedge product in the sum may contain more than two basis 1-forms. The three equations comprise all the possible combinations of two bases out of four dimensions. Reversing the order of bases in a wedge product would only entail a reversal of sign in Ψ_1 and/or Ψ_2.Terms with two identical basis 1-forms go to zero, leaving a 2-form with only one summand. If both terms have two identical basis 1-forms, $\boldsymbol{\Psi} = \boldsymbol{0}$. Adding any number of additional terms with any one of the three available basis wedge products would only require adding their 1-form components into Ψ_1 and/or Ψ_2. Hence, one can always express $\boldsymbol{\Psi}$ as the sum of two wedge products.