Needham, VDGF, Chapter 39, Exercise 7, p. 466.
G. Palmer, July 7, 2026, 1:15 pm PT

7. Factorizes in $\mathbb{R}^3$. In $\mathbb{R}^3$, let $\boldsymbol{\Psi}$ be a general 2-form. Show that $\boldsymbol{\Psi}$ can always be factorized as the wedge product of two 1-forms: $\boldsymbol{\Psi} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.

The 2-form is defined as the wedge product of two 1-forms (34.3).

$$\varphi \wedge \psi = \varphi \otimes \psi - \psi \otimes \varphi$$

A general 2-form in three dimensions must correspond to the wedge product to two general 1-forms such as the following:

$$\boldsymbol{\alpha} = \alpha_1\, \boldsymbol{d}x + \alpha_2\, \boldsymbol{d}y + \alpha_3\, \boldsymbol{d}z$$
$$\boldsymbol{\beta} = \beta_1\, \boldsymbol{d}x + \beta_2\, \boldsymbol{d}y + \beta_3\, \boldsymbol{d}z$$

Then expand the wedge products to tensor products

$$\begin{aligned}\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = {}& (\alpha_1\, \boldsymbol{d}x \otimes \beta_1\, \boldsymbol{d}x - \beta_1\, \boldsymbol{d}x \otimes \alpha_1\, \boldsymbol{d}x) + (\alpha_1\, \boldsymbol{d}x \otimes \beta_2\, \boldsymbol{d}y - \beta_2\, \boldsymbol{d}y \otimes \alpha_1\, \boldsymbol{d}x) + (\alpha_1\, \boldsymbol{d}x \otimes \beta_3\, \boldsymbol{d}z - \beta_3\, \boldsymbol{d}z \otimes \alpha_1\, \boldsymbol{d}x) + \\ & (\alpha_2\, \boldsymbol{d}y \otimes \beta_1\, \boldsymbol{d}x - \beta_1\, \boldsymbol{d}x \otimes \alpha_2\, \boldsymbol{d}y) + (\alpha_2\, \boldsymbol{d}y \otimes \beta_2\, \boldsymbol{d}y - \beta_2\, \boldsymbol{d}y \otimes \alpha_2\, \boldsymbol{d}y) + (\alpha_2\, \boldsymbol{d}y \otimes \beta_3\, \boldsymbol{d}z - \beta_3\, \boldsymbol{d}z \otimes \alpha_2\, \boldsymbol{d}y) + \\ & +(\alpha_3\, \boldsymbol{d}z \otimes \beta_1\, \boldsymbol{d}x - \beta_1\, \boldsymbol{d}x \otimes \alpha_3\, \boldsymbol{d}z) + (\alpha_3\, \boldsymbol{d}z \otimes \beta_2\, \boldsymbol{d}y - \beta_2\, \boldsymbol{d}y \otimes \alpha_3\, \boldsymbol{d}z) + (\alpha_3\, \boldsymbol{d}z \otimes \beta_3\, \boldsymbol{d}z - \beta_3\, \boldsymbol{d}z \otimes \alpha_3\, \boldsymbol{d}z)\end{aligned}$$

After rearranging coefficients, delete all terms with $\boldsymbol{d}x \otimes \boldsymbol{d}x$, $\boldsymbol{d}y \otimes \boldsymbol{d}y$, and $\boldsymbol{d}z \otimes \boldsymbol{d}z$.

$$\begin{aligned}\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = {}& (\alpha_1 \beta_2\, \boldsymbol{d}x \otimes \boldsymbol{d}y - \alpha_1 \beta_2\, \boldsymbol{d}y \otimes \boldsymbol{d}x) + (\alpha_1 \beta_3\, \boldsymbol{d}x \otimes \boldsymbol{d}z - \alpha_1 \beta_3\, \boldsymbol{d}z \otimes \boldsymbol{d}x) + \\ & (\alpha_2 \beta_1\, \boldsymbol{d}y \otimes \boldsymbol{d}x - \alpha_2 \beta_1\, \boldsymbol{d}x \otimes \boldsymbol{d}y) + (\alpha_2 \beta_3\, \boldsymbol{d}y \otimes \boldsymbol{d}z - \alpha_2 \beta_3\, \boldsymbol{d}z \otimes \boldsymbol{d}y) + \\ & +(\alpha_3 \beta_1\, \boldsymbol{d}z \otimes \boldsymbol{d}x - \alpha_3 \beta_1\, \boldsymbol{d}x \otimes \boldsymbol{d}z) + (\alpha_3 \beta_2\, \boldsymbol{d}z \otimes \boldsymbol{d}y - \alpha_3 \beta_2\, \boldsymbol{d}y \otimes \boldsymbol{d}z)\end{aligned}$$

Reduce to wedge products. (Perhaps this and the deletions of like 1-forms could have been done in the first step.)

$$\begin{aligned}\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = {}& \alpha_1 \beta_2\, \boldsymbol{d}x \wedge \boldsymbol{d}y + \alpha_1 \beta_3\, \boldsymbol{d}x \wedge \boldsymbol{d}z + \alpha_2 \beta_1\, \boldsymbol{d}y \wedge \boldsymbol{d}x + \alpha_2 \beta_3\, \boldsymbol{d}y \wedge \boldsymbol{d}z + \\ & +\alpha_3 \beta_1\, \boldsymbol{d}z \wedge \boldsymbol{d}x + \alpha_3 \beta_2\, \boldsymbol{d}z \wedge \boldsymbol{d}y\end{aligned}$$

Find like terms by reversing wedge products and rearrange

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (\alpha_1 \beta_2\, \boldsymbol{d}x \wedge \boldsymbol{d}y + \alpha_2 \beta_1\, \boldsymbol{d}y \wedge \boldsymbol{d}x) + (\alpha_1 \beta_3\, \boldsymbol{d}x \wedge \boldsymbol{d}z + \alpha_3 \beta_1\, \boldsymbol{d}z \wedge \boldsymbol{d}x) + (\alpha_2 \beta_3\, \boldsymbol{d}y \wedge \boldsymbol{d}z + \alpha_3 \beta_2\, \boldsymbol{d}z \wedge \boldsymbol{d}y)$$

$$= (\alpha_1 \beta_2 - \alpha_2 \beta_1)\, \boldsymbol{d}x \wedge \boldsymbol{d}y + (\alpha_1 \beta_3 - \alpha_3 \beta_1)\, \boldsymbol{d}x \wedge \boldsymbol{d}z + (\alpha_2 \beta_3 - \alpha_3 \beta_2)\, \boldsymbol{d}y \wedge \boldsymbol{d}z$$

$$= (\alpha_2 \beta_3 - \alpha_3 \beta_2)\, \boldsymbol{d}y \wedge \boldsymbol{d}z + (\alpha_1 \beta_3 - \alpha_3 \beta_1)\, \boldsymbol{d}x \wedge \boldsymbol{d}z + (\alpha_1 \beta_2 - \alpha_2 \beta_1)\, \boldsymbol{d}x \wedge \boldsymbol{d}y$$

By expanding the wedge product of two general 1-forms into tensor products, one can reduce the result to a general 2-form with three dual bases that are themselves wedge products, showing that the general 2-form can always be written as the wedge product of two 1-forms. The components can be written as the cross product of the coefficients of the two 1-forms, as on p. 380, line -1.

$$\underline{\Psi} = \underline{\boldsymbol{\alpha}} \times \underline{\boldsymbol{\beta}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_2\,\beta_3 - \alpha_3\,\beta_2 \\ \alpha_3\,\beta_1 - \alpha_1\,\beta_3 \\ \alpha_1\,\beta_2 - \alpha_2\,\beta_1 \end{pmatrix}$$