Needham, VDGAF, Chapter 39, Exercise 6, p. 466.
Palmer, June 23, 2026, 1 pm PST

6. Tensor View of Matrix Multiplication as Contraction. Referring back to Section 33.2 and Section 33.6, let **A** be a linear transformation, represented as a vector-valued tensor of valence $\left\{\begin{matrix}1\\1\end{matrix}\right\}$: $\boldsymbol{v} \longrightarrow \boldsymbol{A}(\boldsymbol{v})$. Let B be a second linear transformation, and let **C** be their composition: $\boldsymbol{C}(\boldsymbol{v}) \equiv \boldsymbol{B}[\boldsymbol{A}(\boldsymbol{v})]$. Show that the components of the $\left\{\begin{matrix}1\\1\end{matrix}\right\}$ tensor **C** are given by contraction (aka multiplication): $C^i_j = B^i_k\, A^k_j$.

Notes:

a) To obtain the components in $C^i_j = B^i_k\, A^k_j$, one only need calculate the results for the bases $\boldsymbol{e}_j$ and $\boldsymbol{d}x^i$. See, for example, § **33.6**, first equation, $L^i_j = \boldsymbol{L}\left(\boldsymbol{d}x^i \,||\, \boldsymbol{e}_j\right)$.

b) From **33.2**, a tensor $\boldsymbol{L}(\boldsymbol{\varphi} \,||\, \boldsymbol{v})$ has valence $\left\{\begin{matrix}1\\1\end{matrix}\right\}$. $\boldsymbol{L}(\boldsymbol{v}) \equiv \boldsymbol{L}(\,\ldots \,||\, \boldsymbol{v})$, $\boldsymbol{L}(\boldsymbol{\varphi}) \equiv \boldsymbol{L}(\boldsymbol{\varphi} \,||\, \ldots)$.

c) From **33.6**, $L^i_j\, e_i \otimes \mathrm{d}x^j \Longrightarrow \begin{pmatrix} L^1_j\, v^j \\ L^2_j\, v^j \end{pmatrix} = L^i_j\, v^j = \boldsymbol{L}(\boldsymbol{v})$.

Answer: Proceed by analogy following **33.7**, from top of p. 366. Since there is a $\left\{\begin{matrix}1\\1\end{matrix}\right\}$ tensor $\mathbf{C}(\boldsymbol{\varphi}||\mathbf{v})$, one can write it as

$$\boldsymbol{C}(\boldsymbol{\varphi} \,||\, \boldsymbol{v}) = C^i_j\, \varphi_i\, v^j$$

$$C^i_j = B^i_k\, A^k_j\, \varphi_i\, v^j \quad \text{(Structure of components follows tensor valences.)}$$

$$= \varphi_i\, \boldsymbol{B}\left(\boldsymbol{d}x^i,\, \boldsymbol{e}_k\right) \boldsymbol{A}\left(\boldsymbol{d}x^k,\, \boldsymbol{e}_j\right) v^j \text{ (1-forms in 1-form slots; vectors in vector slots)}$$

$$= \boldsymbol{B}\left(\varphi_i\, \boldsymbol{d}x^i,\, \boldsymbol{e}_k\right) \boldsymbol{A}\left(\boldsymbol{d}x^k,\, v^j\, \boldsymbol{e}_j\right)$$

$$= \boldsymbol{B}\left(\boldsymbol{\varphi},\, \boldsymbol{A}\left(\boldsymbol{d}x^k,\, \boldsymbol{v}\right) \boldsymbol{e}_{k,}\right) \quad \text{(because } \mathbf{A} \text{ is a vector, as in “c)”, above)*}$$

$$= \boldsymbol{B}(\boldsymbol{\varphi},\, \boldsymbol{A}(\,\ldots,\, \boldsymbol{v}))$$

This is a $\left\{\begin{matrix}1\\1\end{matrix}\right\}$ tensor for a fixed **v**. Hence **C** is a $\left\{\begin{matrix}1\\1\end{matrix}\right\}$ tensor with components given by the contraction of $\mathrm{B}^i_k\, \mathrm{A}^k_j$.

*I don’t understand what happens at this point. Evidently one can move $\boldsymbol{d}x^k$ outside the parentheses of **A**() or one can move $\boldsymbol{e}_k$ inside the parentheses. But what does it mean to write $\boldsymbol{d}x^k \boldsymbol{e}_k$?