Needham, VDGF, Chapter 39, Exercise 4, p. 466.
Palmer, August 7, 2026, 4:30 pm PT

4. Matrix Multiplication. *From Schutz (1980). Show that in matrix algebra the action of one matrix on a second matrix, to produce their matrix product, is a tensor of valence* $\left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\}$*. (Hint: Recall that each matrix is itself a tensor of valence* $\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$*; see Section 33.2).*

One can write the matrix as a linear function **L** that takes a 1-form and vector argument, analogous to the example $\boldsymbol{L}(\boldsymbol{\varphi} \,||\, \boldsymbol{v})$ in 33.2. It is not stated in 33.2 that a matrix is itself a tensor of valence $\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$, but we will assume that the exercise hint is correct and write the tensor with 1-form $\boldsymbol{\varphi}$ and vector **u** as $\boldsymbol{M}(\boldsymbol{\varphi} \,||\, \boldsymbol{u})$,. Then we may write a second matrix tensor as $\boldsymbol{N}(\boldsymbol{\psi} \,||\, \boldsymbol{v})$, with 1-form $\boldsymbol{\psi}$ and vector **v**. Now one can write

$$(\boldsymbol{M} \otimes \boldsymbol{N})\,(\boldsymbol{\varphi},\ \boldsymbol{\psi} \,||\, \boldsymbol{u}, \boldsymbol{v}) \equiv \boldsymbol{M}(\boldsymbol{\varphi} \,||\, \boldsymbol{v}) \cdot \boldsymbol{N}(\boldsymbol{\psi} \,||\, \boldsymbol{u}) \qquad (1)$$

Tensor notation seems to leave out some important information. This may be based on the assumption that there is only one way the parts can be assembled into a working formula. But the way is not necessarily obvious. How can one go about contracting $(\boldsymbol{M} \otimes \boldsymbol{N})$ so that it meets the tensor criterion of being real-valued? The LHS of (1) shows by the addition of arguments that $(\boldsymbol{M} \otimes \boldsymbol{N})$ is a tensor of valence $\left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\}$. As explained in 33.2, the action of a matrix on a vector---as in **M**(**v**) or **N**(**u**)--- is another vector. Both of these may be contracted into real values by applying or "inserting" a 1-form. Then the product is also a real value as desired.

$$(\boldsymbol{M} \otimes \boldsymbol{N})\,(\boldsymbol{\varphi},\ \boldsymbol{\psi} \,||\, \boldsymbol{u}, \boldsymbol{v}) \equiv \boldsymbol{\varphi}(\boldsymbol{M}(\boldsymbol{v}))\ \boldsymbol{\psi}(\boldsymbol{N}(\boldsymbol{u}))$$

We have shown that *in matrix algebra the action of one matrix on a second, to produce their matrix product, is a tensor of valence* $\left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\}$. The equation meets the criteria for a tensor of valence $\left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\}$, but is it really matrix multiplication? The matrices are multiplied only indirectly after being contracted first into vectors and subsequently into real numbers? How is one licensed to write $(\boldsymbol{M} \otimes \boldsymbol{N})$? In what physical or geometrical model would one want to profile the matrices by writing "$(\boldsymbol{M} \otimes \boldsymbol{N})$"?

Add a part (ii) to the exercise:

Following the method of Needham on p. 366, let **M** and **N** be tensor matrices of valence $\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$ and let **M** act on **N** by multiplication. Let the components of **M** and **N** be indicated by an upper index i for the 1-form and a lower index j for the vector. Let us prove that the contraction

$$M^i{}_j\, N^k{}_l \equiv C^{\text{ik}}{}_{\text{jl}}$$

yields the components of a new tensor **C** of valence $\left\{\begin{matrix} 2 \\ 2 \end{matrix}\right\}$ such that

$$\boldsymbol{C(\varphi,\ \psi \,||\, u,\ v)} = C^{\text{ik}}{}_{\text{jl}}\, \varphi_i\, \psi_k\, u^j\, v^l$$

To prove this, first observe that the 1-form row vectors φ_i and ψ_k are given the same component indices as their corresponding matrix components, i for $M^i{}_j$ and k for $N^k{}_l$. The same is true for the column vectors, u^j and v^l. Then

$$C^{\text{ik}}{}_{\text{jl}}\, \varphi_i\, \psi_k\, u^j\, v^l = M^i{}_j\, N^k{}_l\, \varphi_i\, \psi_k\, u^j\, v^l$$

$= \left[\varphi_i\, \boldsymbol{M}\left(\boldsymbol{d}x^i,\ \boldsymbol{e}_j\right) u^j\right]\left[\psi_k\, \boldsymbol{N}\left(\boldsymbol{d}x^k,\ \boldsymbol{e}_l\right) v^l\right]$ [by the definition of $\mathtt{M}^{\mathtt{i}}{}_{\mathtt{j}}$ and $N^k{}_l$ and rearrangement of 1-form and vector components]

$= \boldsymbol{M}\left(\varphi_i\, \boldsymbol{d}x^i,\ u^j\, \boldsymbol{e}_j\right) \boldsymbol{N}\left(\psi_k\, \boldsymbol{d}x^k,\ v^l\, \boldsymbol{e}_l\right)$ [by the linearity of tensors]

$= \boldsymbol{M}(\boldsymbol{\varphi},\ \boldsymbol{u})\, \boldsymbol{N}(\boldsymbol{\psi},\ \boldsymbol{v})$ [by the definition of 1-forms and vectors as Einstein sums of components and bases]

$\equiv \boldsymbol{\varphi}(\boldsymbol{M}(\boldsymbol{v}))\, \boldsymbol{\psi}(\boldsymbol{N}(\boldsymbol{u}))$ [by part (i)]

Then from p. 362, ¶ 1 and 2:

$$\equiv (\mathbf{M}\otimes\mathbf{N})\ (\boldsymbol{\varphi},\ \boldsymbol{\psi} \,||\, \mathbf{u},\ \mathbf{v}) = \mathbf{C}(\boldsymbol{\varphi},\ \boldsymbol{\psi} \,||\, \mathbf{u},\ \mathbf{v})$$

We have again shown that *in matrix algebra the action of one matrix on a second matrix, to produce their matrix product, is a tensor of valence* $\left\{\begin{matrix} 2 \\ 2 \end{matrix}\right\}$.

The proposition might also be demonstrated using explicit matrix algebra by expanding $\boldsymbol{M}\left(\varphi_i\, \boldsymbol{d}x^i,\ u^j\, \boldsymbol{e}_j\right) \boldsymbol{N}\left(\psi_k\, \boldsymbol{d}x^k,\ v^l\, \boldsymbol{e}_l\right)$.

$$\boldsymbol{M}\left(\varphi_i\, \boldsymbol{d}x^i,\ u^j\, \boldsymbol{e}_j\right) \boldsymbol{N}\left(\psi_k\, \boldsymbol{d}x^k,\ v^l\, \boldsymbol{e}_l\right) = \left[\varphi_i\, M^i{}_j\, e_j \otimes \mathrm{dx}^i\, u^j\right]\left[\psi_k\, N^k{}_l\, e_l \otimes \mathrm{dx}^k\, v^l\right]$$

$$= \varphi_i\left(M^1{}_1\begin{pmatrix}1\\0\end{pmatrix}(1\times 0) + M^1{}_2\begin{pmatrix}0\\1\end{pmatrix}(1\times 0) + M^2{}_1\begin{pmatrix}1\\0\end{pmatrix}(0\times 1) + M^2{}_2\begin{pmatrix}0\\1\end{pmatrix}(0\times 1)\right) \quad v^l$$

$$u^j\, \psi_k\left(N^1{}_1\begin{pmatrix}1\\0\end{pmatrix}(1\times 0) + N^1{}_2\begin{pmatrix}0\\1\end{pmatrix}(1\times 0) + N^2{}_1\begin{pmatrix}1\\0\end{pmatrix}(0\times 1) + N^2{}_2\begin{pmatrix}0\\1\end{pmatrix}(0\times 1)\right)$$

$$= \varphi_i \left(M^1{}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + M^1{}_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + M^2{}_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + M^2{}_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \qquad v^l$$

$$u^j \, \psi_k \left(N^1{}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + N^1{}_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + N^2{}_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + N^2{}_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \varphi_i \begin{pmatrix} M^1{}_1 & M^1{}_2 \\ M^2{}_1 & M^2{}_2 \end{pmatrix} u^j \, \psi_k \begin{pmatrix} N^1{}_1 & N^1{}_2 \\ N^2{}_1 & N^2{}_2 \end{pmatrix} v^l$$