Needham, VDGF, Chapter 39, Exercise 3, p. 465.
Palmer, May 29, 2026, 2:30 pm PT

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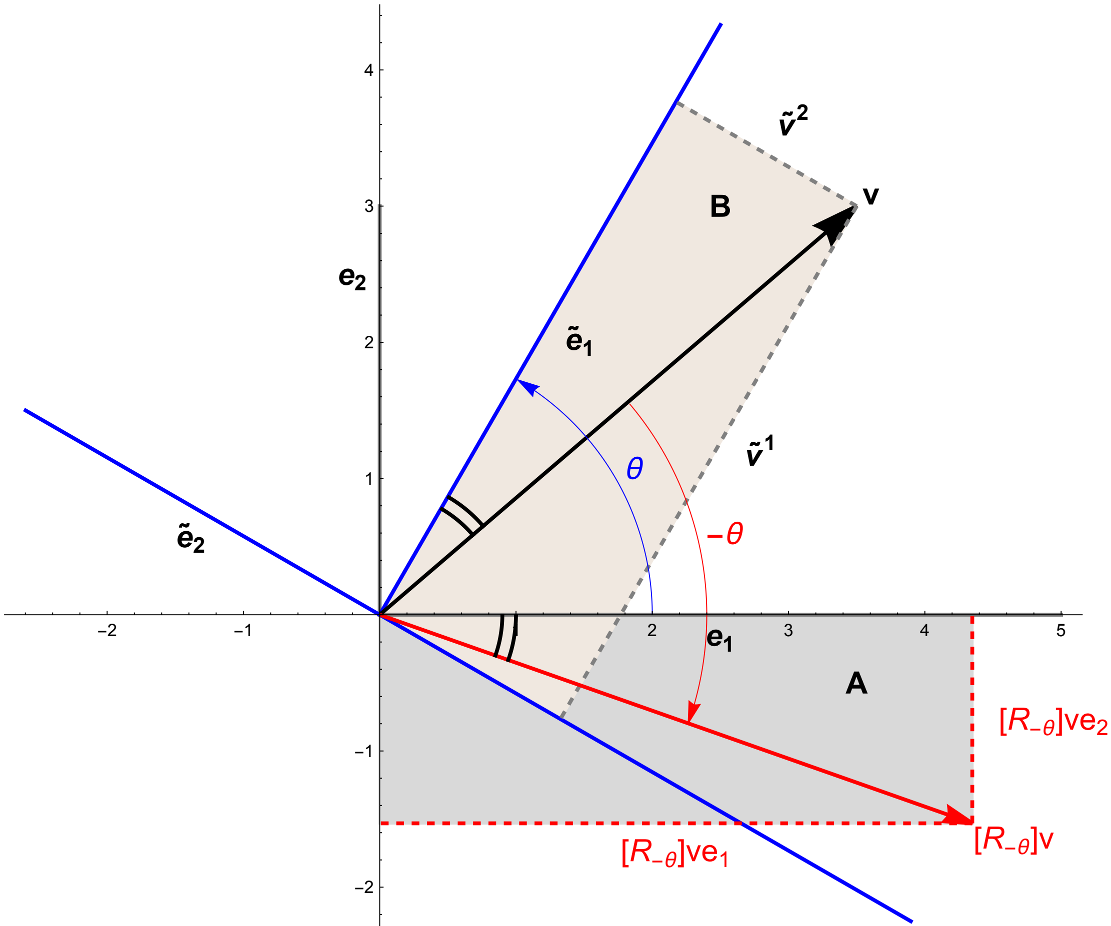

Fig. 1. Demonstration of contravariant rotation of vector v
to obtain components of v on rotated basis $[\tilde{e}_1, \tilde{e}_2]$
acute angles of vector diagonals are equal
by subtraction of Arg(v) from θ and $|-\theta|$

3. Explanation of Covariant and Contravariant. *Let R_θ denote a rotation of $\mathbb{R}^2$ by θ. Then, as explained in [15.3], page 153, its matrix is*

$$[R_\theta] = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \text{ with inverse } [R_\theta]^{-1} = [R_{-\theta}] = [R_\theta]^T = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

where $c \equiv \cos\theta$ and $s \equiv \sin\theta$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard Cartesian basis, and let $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$ be the new basis that results from rotating the old basis by θ, so that $\tilde{\mathbf{e}}_j = R_\theta\, \mathbf{e}_j$.

(i) Draw both bases in the same picture, and also draw a general vector $\mathbf{v} = v^j\, \mathbf{e}_j$, and draw its components $\tilde{v}^j$ in the rotated basis: $\tilde{\mathbf{v}} = \tilde{v}^j\, \tilde{\mathbf{e}}_j$. Use your picture to explain why the new components are obtained by applying the opposite rotation matrix $[R_\theta]^{-1}$ to the original components:

$$\begin{pmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{pmatrix} = [R_\theta]^{-1} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

*Because the components vary in the opposite ("contra-") way to the basis vectors, in the older literature a vector is called **contravariant**.*

Figure 1 demonstrates that components obtained by projecting **v** onto $\{\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2\}$ equal those obtained by projecting $[R_{-\theta}]\,\boldsymbol{v}$ onto $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. The proof rests on the fact that **v** is invariant (held stationary) while the basis vectors are rotated, and equal angles have equal sines. One can also see that magnitudes of vectors are conserved under rotation, so the sides of the rectangles are also conserved. Rectangle B was made by rotating rectangle A by θ.

I think what Needham intended to convey is if one holds a vector constant and rotates the basis, one needs new components to describe the vector within the new basis. The new components can be found easily by rotating just the vector itself by the same degree, but in the opposite direction, making it "contravariant" to the rotating basis vectors.

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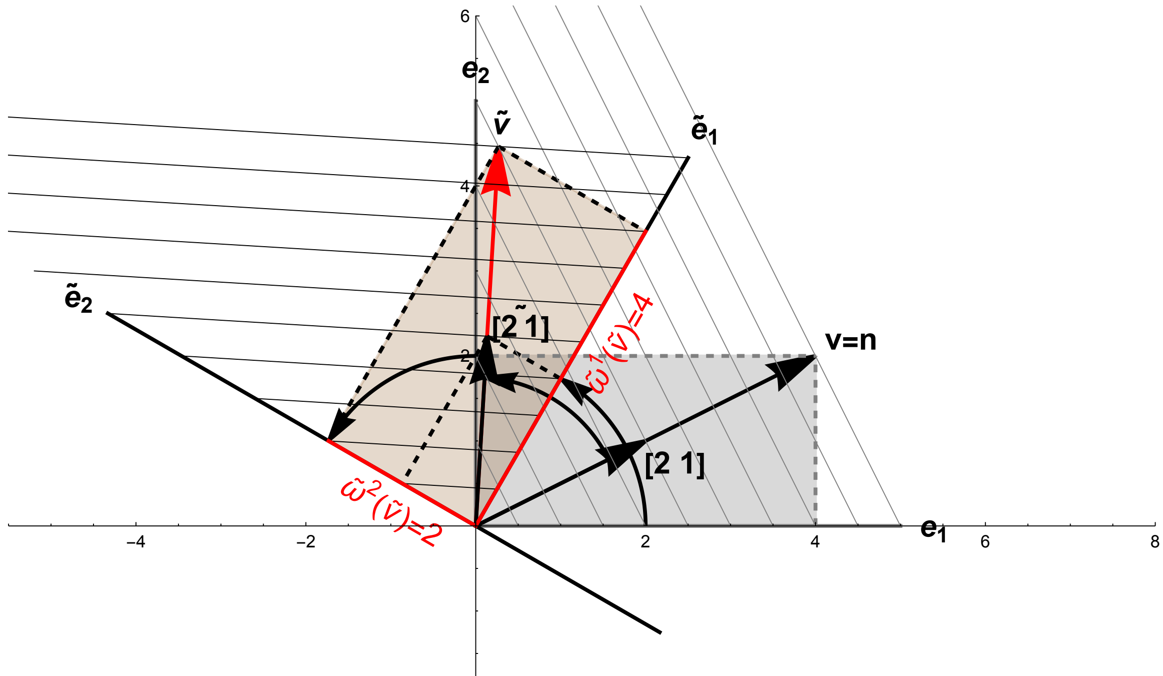

Fig. 2. Covariant components (dotted lines) of 1–form
$\varphi = 2\tilde{\omega}^1 + \tilde{\omega}^2$ based on $\varphi = 2dx + dy$ as in [32.9] (gray)
rotated by θ=Pi/3.
Basis, v, and stack rotate covariantly (θ–rotation arrows).

(ii) The dual basis of $\{\mathbf{e}_1, \mathbf{e}_2\}$ is $\{\mathbf{d}x^1, \mathbf{d}x^2\}$. Let $\varphi = \varphi_j\, \mathbf{d}x^j$ be a general 1-form, and let $\{\tilde{\omega}^1, \tilde{\omega}^2\}$ be the basis duel to $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$, so that the new components are given by $\varphi = \tilde{\varphi}_j\, \tilde{\boldsymbol{\omega}}^j$. Show that the components of the 1-form transform in the same ("co-") way as the basis vectors:

$$\begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = [R_\theta] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

It is important to note that the Dynamic Errata rewrites this equation to be

$$[\tilde{\varphi}_1,\ \tilde{\varphi}_2] = [\varphi_1,\ \varphi_2]\,[R_\theta]$$

In our view, the original equation is correct and should not have been altered. It subjects the 1-form vector components to a positive (anticlockwise) rotation. Hence, the 1-form vector components transform (are transformed) like those of the basis vectors.

The aim of the exercise is to show that the components of the 1-form transform in the same ("co") way as the basis vectors. In this context, the term "transform" strikes me as ambiguous between the passive sense “is transformed (like a vector)” and the active customary sense “transform vectors”. The former passive interpretation seems more common and natural in American English. Perhaps the active interpretation is more common in other English speaking countries. But the components of a 1-form, even when rotated by $[R_\theta]$, do not transform other vectors in any particular direction or magnitude—they only transform them into a real number. In that sense, I suppose one could say that if a 1-form result equals that of another 1-form, they transform in the same way, but that seems a weak interpretation.

Needham wrote: *Let $\varphi = \varphi_j\, \boldsymbol{d}x^j$ be a general 1-form, and let $\{\tilde{\omega}^1,\ \tilde{\omega}^2\}$ be the basis duel to $\{\tilde{\boldsymbol{e}}_1,\ \tilde{\boldsymbol{e}}_2\}$, so that the new components are given by $\varphi = \tilde{\varphi}_j\, \tilde{\omega}^j$.* The exercise statement strongly implies that the components of $\tilde{\varphi}_j$ are defined on the dual basis $\{\tilde{\omega}^1,\ \tilde{\omega}^2\}$. Then supposing that P is a projection function, $\tilde{\omega}^j([\tilde{\varphi}_1,\ \tilde{\varphi}_2]) = P_{\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}}([\tilde{\varphi}_1,\ \tilde{\varphi}_2]) = \tilde{\varphi}_j \bullet \tilde{\boldsymbol{e}}_j$ defines the components $\tilde{\varphi}_j$ on $\tilde{\boldsymbol{e}}_j$.

Using our example from **[32.9]**, it is given that $\boldsymbol{\varphi} = 2\,\boldsymbol{d}x + \boldsymbol{d}y$. We take the 1-form vector to be [2 1]. Let $\theta = \pi/3$, as in Figure 2. We can multiply $[R_{\pi/3}].[2,\ 1]^T$ to rotate the vector as instructed by Needham’s equation, but it is still undefined on $\{\tilde{\boldsymbol{e}}_1,\ \tilde{\boldsymbol{e}}_2\}$.

One could instead multiply by $(-\theta)$ by simply reversing the order of multiplication by writing $[\varphi_1,\ \varphi_2].[R_\theta]$, as in the Dynamic Errata. Then the new components of $\tilde{\varphi}_j$ on $\{\tilde{\boldsymbol{e}}_1,\ \tilde{\boldsymbol{e}}_2\}$ could be read on $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2\}$, but that would be contravariant, contradicting the exercise statement for part (ii).

The method that most clearly demonstrates the covariant approach is to read the new components $[\tilde{\varphi}_1,\ \tilde{\varphi}_2]$. by projecting from the tip of vector $[\tilde{\varphi}_1,\ \tilde{\varphi}_2]$ directly to $\{\tilde{\boldsymbol{e}}_1,\ \tilde{\boldsymbol{e}}_2\}$. Given the whole number components, the new components can easily be evaluated geometrically and visually by dropping a line perpendicular to $\{\tilde{\boldsymbol{e}}_1,\ \tilde{\boldsymbol{e}}_2\}$ from the tip of rotated vector $[\tilde{\varphi}_1,\ \tilde{\varphi}_2]$. Since we have rotated the stack along with the 1-form vector, it easy to read the 1-form vector components in Figure 2 by counting the lines crossed *on the appropriate axis* starting from the origin. The distance between stack lines is 1/2 on $\tilde{\boldsymbol{e}}_1$ and 1 on $\tilde{\boldsymbol{e}}_2$, the same as on $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2\}$.

It is even easier when one considers that simple geometry dictates that the projections of rotated [2, 1] must be half the projections of the chosen arbitrary vector [4, 2] when rotated, so one can write

$$\varphi(v) = [2,\ 1][4,\ 2] = [R_{\pi/3}].[2,\ 1]^T [R_{\pi/3}].[4,\ 2]^T = [2,\ 1]^{\sim}[4,\ 2]^{T\sim} = 10 \qquad (1)$$

where $[a,\ b]^{\sim}$ shows the values of the rotated vector components as defined on (projected onto) the rotated basis.

Since 10 is the desired answer needed to conserve the number of lines pierced by the arbitrary vector on the 1-form stack under rotation of the basis, one can deduce that **v** must rotate with the basis.

Hence, one can say that φ_j, ω^j, and v^j are all covariant with $\boldsymbol{e}_j$. They transform (are transformed) in the same co-way as the basis vectors. In (1), $[4,\ 2]^{T\sim}$ represents $\tilde{\omega}^j(\tilde{\boldsymbol{v}})$. We can write

$$\varphi(\tilde{\boldsymbol{v}}) = [\tilde{\varphi}_1,\ \tilde{\varphi}_1][\tilde{\omega}^1,\ \tilde{\omega}^2](\tilde{\boldsymbol{v}}) = \tilde{\varphi}_j\, \tilde{\omega}^j(\tilde{\boldsymbol{v}})$$

Abstracting away the vectors, one obtains $\varphi = \tilde{\varphi}_j\, \tilde{\omega}^j$, so that the new components are given as prescribed in the exercise.

I think what this says is that the new rotated 1-form stack $\varphi = \tilde{\varphi}_1\, \tilde{\omega}^1 + \tilde{\varphi}_1\, \tilde{\omega}^1$ evaluates on its own terms as though a vector were always in the new coordinates. A vector in the new 1-form has a negatively rotated counterpart in the old basis, and it has doppelganger 1-forms and stacks at all angles.

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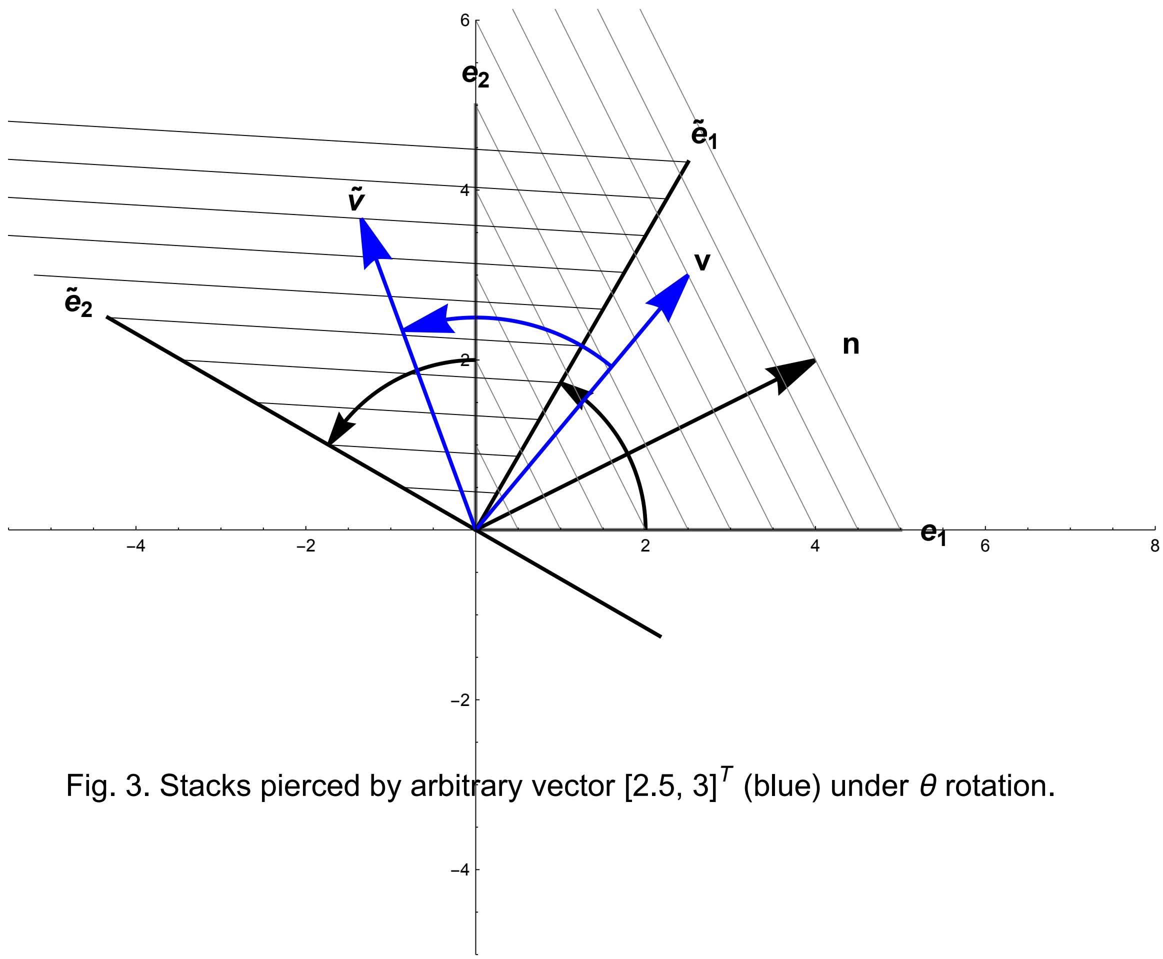

Fig. 3. Stacks pierced by arbitrary vector $[2.5,\ 3]^T$ (blue) under θ rotation.

The equation was tested on the normal value, but it must work on all vectors from the origin to parts of the stack. Figure 3 shows the stacks pierced by $\mathbf{v} = [2.5,\ 3]^T$, with components on the unrotated stack, and $\tilde{\boldsymbol{v}}$ with components on the rotated stack. An example non-normal vector is $\begin{pmatrix} 2.5 \\ 3 \end{pmatrix}$, which by inspection, pierces 8 stack lines, as in Figure 3.

$$[2,\ 1]\left(\tilde{\omega}\,[R_\theta]\right)(\boldsymbol{v})\right) = [2,\ 1]\left(\tilde{\omega}\,[R_\theta]\begin{pmatrix}2.5\\3\end{pmatrix}\right) = [2,\ 1].\begin{pmatrix}2.5\\3\end{pmatrix} = 8$$

Then it is also the case that

$$\varphi(\boldsymbol{v}) = \tilde{\varphi}_j\,\tilde{\omega}^j([R_\theta]\,\boldsymbol{v}) = \tilde{\varphi}_j\,\tilde{\omega}^j(\tilde{\boldsymbol{v}}) = (2\cdot 2.5) + (1\cdot 3) = 8$$

as we wished to show. $\tilde{\varphi}_j$ is tacitly projected onto $\tilde{\omega}$.