

Needham, VDGf, Chapter 39, Exercise 19, p. 471

G. Palmer, Sep 15, 2026, 6 pm, PT

19. Closed and Exact 2-Forms. Verify that each of the following 2-forms is closed. Then show that each is exact, by explicitly finding a 1-form potential ϕ such that $\Psi = d\phi$.

(i) $\Psi = 3x^2y^4 dx \wedge dy$

$$\begin{aligned} d\Psi &= d(3x^2y^4 dx \wedge dy) \\ &= [\partial_x(3x^2y^4) dx + \partial_y(3x^2y^4) dy] \wedge dx \wedge dy \\ &= 6xy^4 dx \wedge dx \wedge dy + 12x^2y^3 dy \wedge dx \wedge dy \\ &= 6xy^4 dx \wedge dx \wedge dy - 12x^2y^3 dx \wedge dy \wedge dy \\ &= 0 - 0 = 0 \quad (\text{because } dx^i \wedge dx^i = 0) \end{aligned}$$

Since $d\Psi = 0$, Ψ is closed.

Let constants of integration be 0.

$$\begin{aligned} \phi &= \left(\int 3x^2y^4 dx \right) dy = 3 \times \frac{1}{3} x^3 y^4 dy = x^3 y^4 dy \\ d\phi &= \partial_x(x^3 y^4) dy + \partial_y(x^3 y^4) dy \\ &= 3x^2 y^4 dx \wedge dy + 4x^3 y^3 dy \wedge dy \\ &= 3x^2 y^4 dx \wedge dy = \Psi \quad (\text{because } dy \wedge dy = 0) \end{aligned}$$

Since $d\phi = \Psi$, Ψ is exact.

(ii) $\Psi = 2xyz^3 dx \wedge dy - 3x^2yz^2 dy \wedge dz$

Use $dx^i \wedge dx^i = 0$.

$$\begin{aligned} d\Psi &= d(2xyz^3 dx \wedge dy) - d(3x^2yz^2 dy \wedge dz) \\ d\Psi &= [\partial_x(2xyz^3) dx + \partial_y(2xyz^3) dy + \partial_z(2xyz^3) dz] \wedge dx \wedge dy \\ &\quad - [\partial_x(3x^2yz^2) dx + \partial_y(3x^2yz^2) dy + \partial_z(3x^2yz^2) dz] \wedge dy \wedge dz \\ &= 0 + 0 + \partial_z(2xyz^3) dz \wedge dx \wedge dy - \partial_x(3x^2yz^2) dx \wedge dy \wedge dz + 0 + 0 \\ &= 6xyz^2 dz \wedge dx \wedge dy - 6xyz^2 dx \wedge dy \wedge dz \\ &= 6xyz^2 dx \wedge dy \wedge dz - 6xyz^2 dx \wedge dy \wedge dz \\ &= 0 \end{aligned}$$

$d\Psi = 0$, so Ψ is closed.

$$\phi = \left(\int 2xyz^3 dx \right) dy - \left(\int 3x^2yz^2 dy \right) dz$$

$$= x^2 y z^3 \, dy - \frac{3}{2} x^2 y^2 z^2 \, dz$$

$$\begin{aligned} d\varphi_1 &= \partial_x (x^2 y z^3) \, dy + \partial_z (x^2 y z^3) \, dy \\ &= 2 x y z^3 \, dx \wedge dy - 3 x^2 y z^2 \, dy \wedge dz \quad (dx^i \wedge dx^i = 0) \end{aligned}$$

This part gives us the original Ψ . See below.

$$\begin{aligned} d\varphi_2 &= \partial_x \left(-\frac{3}{2} x^2 y^2 z^2 \right) \, dz + \partial_y \left(-\frac{3}{2} x^2 y^2 z^2 \right) \, dz \\ &= -3 x y^2 z^2 \, dx \wedge dz - 3 x^2 y z^2 \, dy \wedge dz \end{aligned}$$

This part is not needed.

$$\Psi = 2 x y z^3 \, dx \wedge dy - 3 x^2 y z^2 \, dy \wedge dz$$

$$\text{Therefore, } \varphi = \left(\int 2 x y z^3 \, dx \right) dy = x^2 y z^3 \, dy.$$

Since $d\varphi = \Psi$, Ψ is exact.