

Needham, VDGf, Chapter 39, Exercise 18, p. 471

G. Palmer, Sep 17, 2026, 5:45 pm, PT

18. Closed and Exact 1-Forms. ...Confirm that each of the following is conservative by proving that each is closed: $d\varphi = 0$. Then show that each is exact, by explicitly finding a matching potential energy function, f .

We note that $d\mathbf{f} = \partial_i f d\mathbf{x}^i$ (p. 394).

(i) $\varphi = x d\mathbf{x} + y^2 d\mathbf{y} + z^3 d\mathbf{z}$

$$d\varphi = d(x d\mathbf{x} + y^2 d\mathbf{y} + z^3 d\mathbf{z})$$

$$\begin{aligned} &= \partial_x(x) d\mathbf{x} + \partial_y(x) d\mathbf{x} + \partial_z(x) d\mathbf{x} \\ &+ \partial_x(y^2) d\mathbf{y} + \partial_y(y^2) d\mathbf{y} + \partial_z(y^2) d\mathbf{y} \\ &+ \partial_x(z^3) d\mathbf{z} + \partial_y(z^3) d\mathbf{z} + \partial_z(z^3) d\mathbf{z} \end{aligned}$$

$$\begin{aligned} &= 1 \cdot d\mathbf{x} \wedge d\mathbf{x} + 0 d\mathbf{y} \wedge d\mathbf{x} + 0 d\mathbf{z} \wedge d\mathbf{x} \\ &+ 0 d\mathbf{x} \wedge d\mathbf{y} + 2 y \cdot d\mathbf{y} \wedge d\mathbf{y} + 0 d\mathbf{z} \wedge d\mathbf{y} \\ &+ 0 d\mathbf{x} \wedge d\mathbf{z} + 0 d\mathbf{y} \wedge d\mathbf{z} + 3 z^2 \cdot 0 \end{aligned}$$

$$\begin{aligned} &= 1 \cdot d\mathbf{x} \wedge d\mathbf{x} + 0 d\mathbf{y} \wedge d\mathbf{x} + 0 d\mathbf{z} \wedge d\mathbf{x} \\ &+ 0 d\mathbf{x} \wedge d\mathbf{y} + 2 y \cdot d\mathbf{y} \wedge d\mathbf{y} + 0 d\mathbf{z} \wedge d\mathbf{y} \\ &+ 0 d\mathbf{x} \wedge d\mathbf{z} + 0 d\mathbf{y} \wedge d\mathbf{z} + 3 z^2 \cdot d\mathbf{z} \wedge d\mathbf{z} \end{aligned}$$

$$\begin{aligned} &= 1 \cdot 0 + 0 d\mathbf{y} \wedge d\mathbf{x} + 0 d\mathbf{z} \wedge d\mathbf{x} \\ &+ 0 d\mathbf{x} \wedge d\mathbf{y} + 2 y \cdot 0 + 0 d\mathbf{z} \wedge d\mathbf{y} \\ &+ 0 d\mathbf{x} \wedge d\mathbf{z} + 0 d\mathbf{y} \wedge d\mathbf{z} + 3 z^2 \cdot 0 \end{aligned}$$

$$= 0 + 0 + 0 = 0$$

$d\varphi = 0$, so φ is closed.

Since φ is a 1-form, the potential energy form f must be a 0-form.

$$f = \int x d\mathbf{x} + \int y^2 d\mathbf{y} + \int z^3 d\mathbf{z} = \frac{1}{2} x^2 + \frac{1}{3} y^3 + \frac{1}{4} z^4 + \text{const.}$$

$$df = d\left(\frac{1}{2} x^2\right) + d\left(\frac{1}{3} y^3\right) + d\left(\frac{1}{4} z^4\right)$$

$$= \partial_x\left(\frac{1}{2} x^2\right) d\mathbf{x} + \partial_y\left(\frac{1}{3} y^3\right) d\mathbf{y} + \partial_z\left(\frac{1}{4} z^4\right) d\mathbf{z}$$

$$\begin{aligned}
& +\partial_x\left(\frac{1}{3}y^3\right)dx + \partial_y\left(\frac{1}{3}y^3\right)dy + \partial_z\left(\frac{1}{3}y^3\right)dz \\
& +\partial_x\left(\frac{1}{4}z^4\right)dx + \partial_y\left(\frac{1}{4}z^4\right)dy + \partial_z\left(\frac{1}{4}z^4\right)dz \\
& = xdx + 0dy + 0dz \\
& + 0dx + y^2dy + 0dz \\
& + 0dx + 0dy + z^3dz \\
& = xdx + y^2dy + z^3dz = \varphi
\end{aligned}$$

$$(ii) \varphi = 3x^2y^2zdx + 2x^3yzdy + x^3y^2dz$$

$$\begin{aligned}
d\varphi &= d(3x^2y^2zdx + 2x^3yzdy + x^3y^2dz) \\
&= [\partial_x(3x^2y^2z)dx + \partial_y(3x^2y^2z)dy + \partial_z(3x^2y^2z)dz] \wedge dx \\
&+ [\partial_x(2x^3yz)dy + \partial_y(2x^3yz)dy + \partial_z(2x^3yz)dy] \wedge dy \\
&+ [\partial_x(x^3y^2)dz + \partial_y(x^3y^2)dz + \partial_z(x^3y^2)dz] \wedge dz \\
&= 6xy^2zdx \wedge dx + 6x^2yzdy \wedge dx + 3x^2y^2dz \wedge dx \\
&+ 6x^2yzdx \wedge dy + 2x^3zdy \wedge dy + 2x^3ydz \wedge dy \\
&+ 3x^2y^2dx \wedge dz + 2x^3ydy \wedge dz + 0dz \wedge dz
\end{aligned}$$

Use $dx^i \wedge dx^i = 0$, rearrange order, and change sign of wedge products with like bases. Group results by like bases.

$$\begin{aligned}
&= 0 + 6x^2yzdy \wedge dx + 3x^2y^2dz \wedge dx \\
&+ 6x^2yzdx \wedge dy + 0 + 2x^3ydz \wedge dy \\
&+ 3x^2y^2dx \wedge dz + 2x^3ydy \wedge dz + 0 \\
&= -6x^2yzdx \wedge dy + 6x^2yzdx \wedge dy \\
&+ 3x^2y^2dz \wedge dx - 3x^2y^2dz \wedge dx \\
&- 2x^3ydy \wedge dz + 2x^3ydy \wedge dz \\
&= 0
\end{aligned}$$

$d\varphi = 0$, so φ is closed.

If the integrals of one or more of dx , dy , dz result in the same function, then it only appears once in φ . Assume zero constants of integration, which would in any case have no effect on φ .

$$\varphi = 3x^2y^2zdx + 2x^3yzdy + x^3y^2dz$$

$$\int 3x^2 y^2 z \, dx = x^3 y^2 z$$

$$\int 2x^3 yz \, dy = x^3 y^2 z$$

$$\int x^3 y^2 \, dz = x^3 y^2 z$$

$$f = x^3 y^2 z$$

$$df = \partial_x(x^3 y^2 z) dx + \partial_y(x^3 y^2 z) dy + \partial_z(x^3 y^2 z) dz = \varphi$$

$$df = 3x^2 y^2 z dx + 2x^3 yz dy + x^3 y^2 dz = \varphi$$

$$(iii) \varphi = (2xy + z) dx + (x^2 + 3y^2) dy + x dz$$

$$d\varphi = d[(2xy + z) dx + (x^2 + 3y^2) dy + x dz]$$

$$= d(2xy + z) dx + d(x^2 + 3y^2) dy + d(x) dz$$

$$\begin{aligned} &= \partial_x(2xy + z) dx + \partial_y(2xy + z) dy + \partial_z(2xy + z) dz \\ &+ \partial_x(x^2 + 3y^2) dy + \partial_y(x^2 + 3y^2) dy + \partial_z(x^2 + 3y^2) dy \\ &+ \partial_x(x) dz + \partial_y(x) dz + \partial_z(x) dz \end{aligned}$$

$$\begin{aligned} &= 2y dx \wedge dx + 2x dy \wedge dx + dz \wedge dx \\ &+ 2x dx \wedge dy + 6y dy \wedge dy + 0 dz \wedge dy \\ &+ dx \wedge dz + 0 dy \wedge dz + 0 dz \wedge dz \end{aligned}$$

$$\begin{aligned} &= 0 + 2x dy \wedge dx + dz \wedge dx \\ &+ 2x dx \wedge dy + 0 + 0 \\ &+ dx \wedge dz + 0 + 0 \end{aligned}$$

$$\begin{aligned} &= -2x dx \wedge dy + 2x dx \wedge dy + dz \wedge dx - dz \wedge dx \\ &= 0 \end{aligned}$$

The result is $d\varphi = 0$, so φ is closed.

If the integrals of one or more of dx , dy , dz result in the same function, then it only appears once in f .

$$\varphi = (2xy + z) dx + (x^2 + 3y^2) dy + x dz$$

$$\int (2xy + z) dx = x^2 y + zx$$

$$\int (x^2 + 3y^2) dy = x^2 y + y^3$$

$$\int x \, dz = xz$$

$$f = x^2 y + y^3 + xz \quad (\text{even though } x^2 y \text{ and } xz \text{ appear in two integrals})$$

$$df = (2xy + z) \, dx + (x^2 \, dy + 3y^2 \, dy) + x \, dz = \varphi$$