

Needham, VDGf, Chapter 39, Exercise 17, p. 470

G. Palmer, Sep 13, 2026, 12:15 pm, PT

17. *Spinning Photons and Self-Duality. In Minkowski spacetime, a complex 2-form Ψ is called self-dual if*

$$\star \Psi = i\Psi$$

and anti-self-dual if

$$\star \Psi = -i\Psi,$$

where $\star$ is the Hodge duality operator introduced in Exercise 15.

(i) Let us verify that the Farady 2-form F (or any 2-form, for that matter) can always be split into a self-dual part ${}^+F$ and an anti-self-dual part ${}^-F$, so that

$$F = {}^+F + {}^-F.$$

Starting from this formula, and using the fact that $\star {}^\pm F = \pm i {}^\pm F$, deduce that

$${}^+F = \frac{1}{2}[F - i\star F] \quad \text{and} \quad {}^-F = \frac{1}{2}[F + i\star F]. \quad (\text{See the dynamic errata on p. 470})$$

Deduce ${}^+F = \frac{1}{2}[F - i\star F]$ and ${}^-F = \frac{1}{2}[F + i\star F]$.

I do not understand why multiplying the Hodge dual by $\pm i$ creates asymmetry. A geometric explanation would be useful. Since Faraday is a 2-form, one might view it as the signed area of a parallelogram or the quantity of a current normal to a surface. But what is the function of the imaginary i ? Does that introduce the complex plane for F and $\star F$?

$$F = {}^+F + {}^-F \quad (1) \quad \text{the decomposition}$$

$$\star {}^+F = i {}^+F \quad (3) \quad \text{from the exercise}$$

$$\star {}^-F = -i {}^-F \quad (4)$$

Then

$$\star F = \star ({}^+F + {}^-F) = \star {}^+F + \star {}^-F$$

$$\star F = i {}^+F - i {}^-F \quad \text{substitute (3) and (4)}$$

Solve for ${}^-F$

$$i\star F = -{}^+F + {}^-F \quad \text{multiply both sides by } i$$

$$F + i\star F = (-{}^+F + {}^-F) + ({}^+F + {}^-F) \quad \text{add } F \text{ to both sides}$$

$$2{}^-F = F + i\star F \quad \text{swap sides of equation}$$

$${}^-F = \frac{1}{2}[F + i\star F]$$

Solve for ${}^+F$

$$-i\star F = {}^+F - {}^-F \quad \text{multiply both sides by } -i$$

$$F - i\star F = ({}^+F - {}^-F) + ({}^+F + {}^-F) \quad \text{add } F \text{ to both sides}$$

$${}^+F = F - i\star F \quad \text{swap sides of equation}$$

$${}^+F = \frac{1}{2}[F - i\star F]$$

One can show that the equations are valid for $\mathbf{F} = \boldsymbol{\epsilon} \wedge \mathbf{dt} + \mathbf{B}$. See pp. 383 and 469 for essential equations. All the basis 1-forms are external derivatives, normally written with boldface $\mathbf{d}$.

$$\star \mathbf{F} = \boldsymbol{\beta} \wedge \mathbf{dt} - \mathbf{E} \quad (\text{p. 469})$$

$$\boldsymbol{\epsilon} \wedge \mathbf{dt} = E_x \mathbf{dx} \wedge \mathbf{dt} + E_y \mathbf{dy} \wedge \mathbf{dt} + E_z \mathbf{dz} \wedge \mathbf{dt}$$

$$\mathbf{B} = B_x \mathbf{dy} \wedge \mathbf{dz} + B_y \mathbf{dz} \wedge \mathbf{dx} + B_z \mathbf{dx} \wedge \mathbf{dy}$$

$$\boldsymbol{\beta} = B_x \mathbf{dx} + B_y \mathbf{dy} + B_z \mathbf{dz}$$

$$\begin{aligned} {}^+ \mathbf{F} &= \frac{1}{2} [(\boldsymbol{\epsilon} \wedge \mathbf{dt} + \mathbf{B}) - i(\boldsymbol{\beta} \wedge \mathbf{dt} - \mathbf{E})] \\ &= \frac{1}{2} [(E_x \mathbf{dx} \wedge \mathbf{dt} + E_y \mathbf{dy} \wedge \mathbf{dt} + E_z \mathbf{dz} \wedge \mathbf{dt} + B_x \mathbf{dy} \wedge \mathbf{dz} + B_y \mathbf{dz} \wedge \mathbf{dx} + B_z \mathbf{dx} \wedge \mathbf{dy}) \\ &\quad - i(B_x \mathbf{dx} \wedge \mathbf{dt} + B_y \mathbf{dy} \wedge \mathbf{dt} + B_z \mathbf{dz} \wedge \mathbf{dt} - E_x \mathbf{dy} \wedge \mathbf{dz} - E_y \mathbf{dz} \wedge \mathbf{dx} - E_z \mathbf{dx} \wedge \mathbf{dy})] \\ &= \frac{1}{2} [(E_x - iB_x) \mathbf{dx} \wedge \mathbf{dt} + (E_y - iB_y) \mathbf{dy} \wedge \mathbf{dt} + (E_z - iB_z) \mathbf{dz} \wedge \mathbf{dt}) \\ &\quad + (B_x + iE_x) \mathbf{dy} \wedge \mathbf{dz} + (B_y + iE_y) \mathbf{dz} \wedge \mathbf{dx} + (B_z + iE_z) \mathbf{dx} \wedge \mathbf{dy}] \end{aligned}$$

$$\begin{aligned} {}^- \mathbf{F} &= \frac{1}{2} [(\boldsymbol{\epsilon} \wedge \mathbf{dt} + \mathbf{B}) + i(\boldsymbol{\beta} \wedge \mathbf{dt} - \mathbf{E})] \\ {}^- \mathbf{F} &= \frac{1}{2} [(E_x \mathbf{dx} \wedge \mathbf{dt} + E_y \mathbf{dy} \wedge \mathbf{dt} + E_z \mathbf{dz} \wedge \mathbf{dt} + B_x \mathbf{dy} \wedge \mathbf{dz} + B_y \mathbf{dz} \wedge \mathbf{dx} + B_z \mathbf{dx} \wedge \mathbf{dy}) \\ &\quad + i(B_x \mathbf{dx} \wedge \mathbf{dt} + B_y \mathbf{dy} \wedge \mathbf{dt} + B_z \mathbf{dz} \wedge \mathbf{dt} - E_x \mathbf{dy} \wedge \mathbf{dz} - E_y \mathbf{dz} \wedge \mathbf{dx} - E_z \mathbf{dx} \wedge \mathbf{dy})] \\ &= \frac{1}{2} [(E_x + iB_x) \mathbf{dx} \wedge \mathbf{dt} + (E_y + iB_y) \mathbf{dy} \wedge \mathbf{dt} + (E_z + iB_z) \mathbf{dz} \wedge \mathbf{dt}) \\ &\quad + (B_x - iE_x) \mathbf{dy} \wedge \mathbf{dz} + (B_y - iE_y) \mathbf{dz} \wedge \mathbf{dx} + (B_z - iE_z) \mathbf{dx} \wedge \mathbf{dy}] \end{aligned}$$

$$\begin{aligned} \mathbf{F} &= {}^+ \mathbf{F} + {}^- \mathbf{F} = \frac{1}{2} [(2 E_x \mathbf{dx} \wedge \mathbf{dt} + 2 E_y \mathbf{dy} \wedge \mathbf{dt} + 2 E_z \mathbf{dz} \wedge \mathbf{dt}) \\ &\quad + (2 B_x \mathbf{dy} \wedge \mathbf{dz} + 2 B_y \mathbf{dz} \wedge \mathbf{dx} + 2 B_z \mathbf{dx} \wedge \mathbf{dy})] \\ &= \boldsymbol{\epsilon} \wedge \mathbf{dt} + \mathbf{B} \quad [\text{cancel 2 in numerator and denominator and substitute from above.}] \end{aligned}$$

(ii) ... Verify that all four of Maxwell's Equations can be encapsulated into a single complex equation:

$$\mathbf{d} \star \mathbf{F} = -2 \pi i \star \mathbf{J}. \quad (\text{Note again, the error in the formula for } {}^+ \mathbf{F}.)$$

Short form:

$$\begin{aligned} \mathbf{d} \star \mathbf{F} &= \mathbf{d} \left(\frac{1}{2} [\mathbf{F} - i \star \mathbf{F}] \right) \\ &= \frac{1}{2} [\mathbf{dF} - \mathbf{d}(i \star \mathbf{F})] \\ &= \frac{1}{2} [0 - i 4 \pi \star \mathbf{J}] \quad (\text{From pp. 401, 470, Ex. 16(iii) and } \mathbf{d} \star \mathbf{F} = 4 \pi \star \mathbf{J}, \text{ p. 403}) \\ &= -2 \pi i \star \mathbf{J} \end{aligned}$$

Note that $\mathbf{dF}$ uses source-free $\nabla \cdot \mathbf{B}$ and $(\nabla \times \mathbf{E} + \partial_t \mathbf{B})$, which both equal zero (p. 401) and source-based $\mathbf{d} \star \mathbf{F}$ uses $\mathbf{d} \star \mathbf{F} = 4 \pi \star \mathbf{J}$ (p. 403). $\star \mathbf{J} = -\rho \mathbf{V} + \text{flux 2 - form of } \mathbf{j} \wedge \mathbf{dt}$ (36.18), thereby involving ρ from the third equation.

Long form:

$$\begin{aligned}
 \mathbf{d} \star \mathbf{F} &= \mathbf{d} \left(\frac{1}{2} [\mathbf{F} - i \star \mathbf{F}] \right) = \left(\frac{1}{2} [\mathbf{dF} - \mathbf{d} (i \star \mathbf{F})] \right) \\
 &= \frac{1}{2} [(\nabla \cdot \mathbf{B}) \mathbf{V} + \{\text{flux 2 - form of } [\nabla \times \mathbf{E} + \partial_t \mathbf{B}]\} \wedge \mathbf{dt}] - i \mathbf{d} \star \mathbf{F} \quad (\text{p. 401}) \\
 &= \frac{1}{2} [(0 \mathbf{V} + 0 \wedge \mathbf{dt}) - 4 \pi i \{-\rho \mathbf{V} + \text{flux 2 - form of } \mathbf{j}\} \wedge \mathbf{dt}] \quad (\text{p. 402}) \\
 &= \frac{1}{2} [0 - i 4 \pi \star \mathbf{J}] \quad (36.18) \\
 &= -2 \pi i \star \mathbf{J}
 \end{aligned}$$