Needham, VDGF, Chapter 39, Exercise 16, p. 470
G. Palmer, Sep 3, 2026, 9:30 pm, PT

(i) Assuming that electric charge is ***conserved****, i.e., that it can neither be created nor destroyed, deduce that if V is the interior of a closed surface* $S = \partial V$*, then*

$$\frac{d}{dt}\iiint_V \rho \, dV = -\iint_S \boldsymbol{j}\bullet \boldsymbol{n} \, dA.$$

On p. 402, Needham stated that ρ is charge density and **j** is current density (underlined in Needham). ρ and **j** are seemingly listed in the text as *sources*, but the antecedent is actually "fields".

The two sides of the equation must represent equal momentaneous rates of change in charge. On the left hand side, one imagines change in charge within a fixed volume of space. It is a momentaneous change with respect to time.

On the right hand side, one imagines flow of current across the surface of the volume. The current density **j** (a 1-form in three spatial dimensions) is projected onto the normal to the surface of the volume, giving the current, which is equivalent to the momentaneous rate of change of charge with respect to time.

Since the rate of change within the volume equals the rate of change across the surface, the equation satisfies the constraint of conservation of charge.

(ii) Use Gauss's Theorem [(37.14)] *to deduce that the local expression of charge concentration is*

$$\frac{d}{dt}\rho + \nabla \bullet \boldsymbol{j} = 0.$$

ρ is charge density. ***j*** *(underlined in Needham) is current density.*

From (37.14),

$$\iiint_V \nabla \bullet \underline{\psi} \, dV = \iint_S \underline{\psi} \bullet \hat{\boldsymbol{n}} \, dA \Longrightarrow \iiint_V \nabla \bullet \boldsymbol{j} \, dV = \iint_S \boldsymbol{j} \bullet \hat{\boldsymbol{n}} \, dA,$$

because $\underline{\psi}$ and **j** are both vectors.

From (i),

$$\frac{d}{dt}\iiint_V \rho \, dV + \iiint_V \nabla \bullet \boldsymbol{j} \, dV = \iiint_V \left(\frac{d}{dt}\rho\right) dV + \iiint_V (\nabla \bullet \boldsymbol{j}) \, dV = 0$$

$$\Longrightarrow \quad \frac{d}{dt}\rho + \nabla \cdot \boldsymbol{j} = 0$$

Assume the constants of integration equal zero. Since the integrals are equivalent and we assume their ranges are equal, the integrands must be equal in magnitude and opposite in sign. Here, the charge (and its rate of change) takes a negative value, so that the rate wrt time of flux out of the shrinking parallelopipeds symbolized by dV is equaled by the rate of flux of charge density symbolized by $\nabla \cdot \boldsymbol{j}$ out of the shrinking parallelopipeds.

(iii) By taking the exterior derivative of the second pair of Maxwell's Equations, in the form

$$\boldsymbol{d} * \boldsymbol{F} = 4\,\pi * \boldsymbol{J},$$

deduce that charge conservation is a logical consequence of Maxwell's Equations, taking the form

$$\boldsymbol{d} * \boldsymbol{J} = 0.$$

We might attempt to derive $\frac{d}{dt}\rho + \nabla \cdot \boldsymbol{j} = 0$ from an expanded version of $\boldsymbol{d}(\boldsymbol{d} * \boldsymbol{F}) = \boldsymbol{d}(4\,\pi * \boldsymbol{J})$.

$$* \boldsymbol{J} = -\rho \boldsymbol{V} + [\text{Flux } 2 - \text{form of } \mathbf{j}] \wedge \boldsymbol{d}t \quad \text{(p. 402)}$$

$$* \boldsymbol{J} = -\rho \boldsymbol{V}_3 + \{\text{flux } 2 - \text{form of} \mathbf{j}\} \wedge \boldsymbol{d}t \quad \text{(p. 469)}$$

It appears that both $\boldsymbol{V}$ and $\boldsymbol{V}_3$ refer to $\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z$.

$$* \boldsymbol{F} = \beta \wedge \boldsymbol{d}t - \boldsymbol{E} \quad \text{(p. 469)}$$

F disappears in this derivation.

$$\boldsymbol{d} * \boldsymbol{F} = 4\,\pi[-\rho \boldsymbol{V} + \{\text{flux } 2 - \text{form of } \mathbf{j}\} \wedge \boldsymbol{d}t$$

$$\boldsymbol{d}(\boldsymbol{d} * \boldsymbol{F}) = 0 \quad \text{because } \boldsymbol{d}^2 = 0$$

Expand the RHS: $4\,\pi * \boldsymbol{J}$.

$$0 = 4\,\pi[-\rho \boldsymbol{V}_3 + \{\text{flux } 2 - \text{form of} \boldsymbol{j}\} \wedge \boldsymbol{d}t] = -\rho \boldsymbol{V}_3 + \{\text{flux } 2 - \text{form of} \boldsymbol{j}\} \wedge \boldsymbol{d}t$$

$$\rho \boldsymbol{V}_3 - \{\text{flux } 2 - \text{form of} \boldsymbol{j}\} \wedge \boldsymbol{d}t = 0 \quad \text{[Rearrange]}$$

Take the exterior derivative of both sides.

$$\mathbf{d}(\rho \boldsymbol{V}_3) - \boldsymbol{d}(\{\text{flux 2 – form of } \boldsymbol{j}\} \wedge \boldsymbol{d}t) = 0 \quad [\text{Same as } \boldsymbol{d} * \boldsymbol{J} = 0]$$

$$\boldsymbol{d}(\rho \boldsymbol{V}) = \partial_t \rho\, \boldsymbol{d}t \wedge \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = -\partial_t \rho\, \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t = -\partial_t \rho\, \boldsymbol{V}_3 \wedge \boldsymbol{d}t$$

$[\text{all terms with } \boldsymbol{d}x^i \wedge \boldsymbol{d}x^i = 0]$

$$\{\text{flux 2 – form of } \boldsymbol{j}\} \wedge \boldsymbol{d}t = j^1\, \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + j^2\, \boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}t + j^3\, \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t$$

Take the external derivative of $\{\text{flux 2 – form of } \boldsymbol{j}\} \wedge \boldsymbol{d}t$:

$$\boldsymbol{d}\left(j^1\, \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t - j^2\, \boldsymbol{d}x \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + j^3\, \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t\right) \quad [\text{same as } \boldsymbol{d} * \mathbf{j}]$$

$$= \partial_x\left(j^1\right) \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + \partial_y\left(-j^2\right) \boldsymbol{d}y \wedge \boldsymbol{d}x \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + \partial_z\left(j^3\right) \boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t$$

$$= \partial_x\left(j^1\right) \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + \partial_y\left(j^2\right) \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + \partial_z\left(j^3\right) \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t$$

$$= (\nabla \bullet \boldsymbol{j})\, V_3 \wedge \boldsymbol{d}t$$

Now, dividing through by $(-\boldsymbol{V}_3 \wedge \boldsymbol{d}t)$, one can write:

$$-\partial_t \rho\, \boldsymbol{V}_3 \wedge \boldsymbol{d}t - (\nabla \bullet \boldsymbol{j})\, \boldsymbol{V}_3 \wedge \boldsymbol{d}t \implies \frac{d}{dt} \rho + \nabla \bullet \boldsymbol{j} = 0$$

Which is what we wanted.