Needham, VDGF, Chapter 39, Exercise 15, p. 468.
G. Palmer, August 13, 2026, 3 pm, PT

15. Hodge Star Duality Operator (*). ...

(i) in n dimensions, show that the space of p-forms has the same dimension as the space of (n-p)-forms....

In the following $dx^i = \boldsymbol{d}x^i$
The n-p form bases use the dimensions unused by all possible combinations of p-forms.

Try a concrete example, ignoring sign: In $\mathbb{R}^4$, n = 4. In the space of 3-forms, *p* = 3. (n-p) forms are 1-forms. The *-forms are

$\ast dw = dx \wedge dy \wedge dz$
$\ast dx = dw \wedge dy \wedge dz$
$\ast dy = dw \wedge dx \wedge dz$
$\ast dz = dw \wedge dx \wedge dy$

There are four possible p-forms and four (n-p) forms, so they have same dimension.

In $\mathbb{R}^4$, in the space of 2-forms, *p* = 2. (n-p) forms are 2-forms. The *-forms are

$dy \wedge dz = \ast(dt \wedge dx)$
$dx \wedge dz = \ast(dt \wedge dy)$
$dx \wedge dy = \ast(dt \wedge dz)$
$dt \wedge dz = \ast(dx \wedge dy)$
$dt \,\hat{}\, dy = \ast(dx \wedge dz)$
$dt \wedge dx = \ast(dy \wedge dz)$

There are six possible p-forms and six (n-p) forms, so again, they have the same dimension.

By the definition of the Hodge Start Duality Operator (*), there is a *-form for every p-form.

*(ii) In $\mathbb{R}^3$, let $\mathcal{V}_3 = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z$ be the volume 3-form. Given a basis p-form $\boldsymbol{\sigma}$, we define * to be the linear operator that fills in the "missing pieces" of $\mathcal{V}_3$:*

$$\boldsymbol{\sigma} \wedge \ast\boldsymbol{\sigma} = \mathcal{V}_3$$

Deduce that

$\ast\boldsymbol{d}x = \boldsymbol{d}y \wedge \boldsymbol{d}z$
$\ast\boldsymbol{d}y = \boldsymbol{d}z \wedge \boldsymbol{d}x$ (next form after y is **d**z; next after **d**z is **d**x)

$\star \boldsymbol{d}z = \boldsymbol{d}x \wedge \boldsymbol{d}y$ (next form after **d**z is **d**x)

The answer to deducing the $\star$-forms seems obvious, as they as the “missing parts” of $\mathcal{V}_3$ and they retain their volume 3-form order. However, one needs to begin collecting a $\star$–form with the first part after the missing part (e.g. **d**y after $\star \boldsymbol{d}x$) and follow the convention that **d**z precedes **d**x.

Alternatively one can proceed as in (v) and multiply and rearrange to obtain the volume 3-form in $\mathbb{R}^3$.

(iii) Continuing in $\mathbb{R}^3$, we remind you of our earlier notation: $\underline{\varphi}$ is the vector corresponding ($\rightleftharpoons$) to the 1-form $\boldsymbol{\varphi}$. Show that the Hodge duality yields two forms of the vector (cross) product:

$$\underline{\alpha} \times \underline{\beta} = \underline{\gamma} \iff \boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \star \boldsymbol{\gamma} \iff \star \boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{\gamma}$$

The exercise seems to have omitted a key piece of information regarding the Hodge Dual. The star appears to apply only to the basis forms, with no effect on the form components, which can therefore be moved inside or outside the star.

Let $\underline{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \underline{\alpha} \times \underline{\beta}$.

Rewrite (from p. 390),

$$\star \gamma = \star\left(\gamma_1 \boldsymbol{d}x^1 + \gamma_2 \boldsymbol{d}x^2 + \gamma_3 \boldsymbol{d}x^3\right)$$

$$= \gamma_1 \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + \gamma_2 \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + \gamma_3 \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2$$

$$= (a_2 b_3 - a_3 b_2)\, \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + (a_3 b_1 - a_1 b_3)\, \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + (a_1 b_2 - a_2 b_1)\, \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2$$

$$= \boldsymbol{\alpha} \wedge \boldsymbol{\beta} \iff \underline{\alpha} \times \underline{\beta} = \underline{\gamma}$$

$$\underline{\alpha} \times \underline{\beta} = \underline{\gamma} \iff \star \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$$

$$= \star\left[(a_2 b_3 - a_3 b_2)\, \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + (a_3 b_1 - a_1 b_3)\, \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + (a_1 b_2 - a_2 b_1)\, \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2\right]$$

$$= (a_2 b_3 - a_3 b_2) \star\left(\boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3\right) + (a_3 b_1 - a_1 b_3) \star\left(\boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1\right) + (a_1 b_2 - a_2 b_1) \star\left(\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2\right)$$

$$= \gamma_1 \boldsymbol{d}x^1 + \gamma_2 \boldsymbol{d}x^2 + \gamma_3 \boldsymbol{d}x^3 = \boldsymbol{\gamma}$$

(iv) Still in $\mathbb{R}^3$, recall that every 2-form $\boldsymbol{\Psi}$ can be thought of as the flux associated with a flow of velocity $\underline{\Psi}$, via (34.10):

$$\boldsymbol{\Psi} = \Psi^1(\boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3) + \Psi^2(\boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1) + \Psi^3(\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2) \rightleftharpoons \underline{\Psi} = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}$$

If $\boldsymbol{\psi}$ is the 1-form corresponding to $\underline{\Psi}$, deduce that the correspondence between $\boldsymbol{\Psi}$ and $\boldsymbol{\psi}$ is Hodge duality:

$$\star\boldsymbol{\Psi} = \boldsymbol{\psi} \quad \text{and } \boldsymbol{\Psi} = \star\boldsymbol{\psi}.$$

If the correspondence between $\boldsymbol{\Psi}$ and $\boldsymbol{\psi}$ is Hodge duality, then one can write:

$$\boldsymbol{\psi} = \star\boldsymbol{\Psi} = \star\left[\Psi^1(\boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3) + \Psi^2(\boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1) + \Psi^3(\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2)\right]$$

$$= \Psi^1\, \boldsymbol{d}x^1 + \Psi^2\, \boldsymbol{d}x^2 + \Psi^3\, \boldsymbol{d}x^3$$

as with the electrical and magnetic fields on p. 383.

(v) Now consider Minkowski spacetime, with metric

$$ds^2 = g_{ij}\, dx^i\, dx^j = -dt^2 + dx^2 + dy^2 + dz^2.$$

...

Let the volume 4-form be $\boldsymbol{V}_4 = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t$ and define $\star\boldsymbol{d}x^i$ by

$$\boldsymbol{d}x^i \wedge \star\boldsymbol{d}x^i = g_{ij}\, \boldsymbol{V}_4$$

Deduce that

$$\star\boldsymbol{d}t = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = \boldsymbol{V}_3$$
$$\star\boldsymbol{d}x = \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t$$
$$\star\boldsymbol{d}y = \boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}t$$
$$\star\boldsymbol{d}z = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t$$

Let $\boldsymbol{d}x^i \in D = \{\boldsymbol{d}x, \boldsymbol{d}y, \boldsymbol{d}z, \boldsymbol{d}t\}$.

Then $\star\boldsymbol{d}x^i$ = the wedge product of the other three 1-form bases, $D\backslash\boldsymbol{d}x^i$. Multiplying $\boldsymbol{d}x^i \wedge \star\boldsymbol{d}x^i$ completes the four volume up to order and sign. In other words, the three wedged forms on the RHS of each equation complement the starred form on the LHS by filling in the “missing pieces.”

The order matters, because moving a basis $\mathbf{d}x^i$ by one slot (swapping adjacent bases) reverses the sign. Let $g_{ij} = [-1, 1, 1, 1]$. If $g_{ij}\,\boldsymbol{V}_4$ can be obtained by calculating $\boldsymbol{d}x^i \wedge {*}\boldsymbol{d}x^i$ and rearranging the order of bases, then the four equations are valid.

$*\boldsymbol{d}t = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = \boldsymbol{V}_3$

$\boldsymbol{d}t \wedge \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = -\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t = -\boldsymbol{V}_4 = g_{ij}\,\boldsymbol{V}_4$

(**d**t swaps right three times)

$*\boldsymbol{d}x = \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t$

$\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t = \boldsymbol{V}_4 = g_{ij}\,\boldsymbol{V}_4$

(no rearrangement)

$*\boldsymbol{d}y = \boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}t$

$\boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}t = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t = \boldsymbol{V}_4 = g_{ij}\,\boldsymbol{V}_4$

(**d**x swaps left twice)

$*\boldsymbol{d}z = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t$

$\boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t = \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t = \boldsymbol{V}_4 = g_{ij}\,\boldsymbol{V}_4$

(**d**z swaps right twice)

Under the Minkowski metric in which $g_{ij} = [-1, 1, 1, 1]$, the procedure of $\boldsymbol{d}x^i \wedge {*}\boldsymbol{d}x^i$ with rearrangement of bases confirms the three equations.

Why is it that the components of the Minkowski metric are needed to write $g_{ij}\,\boldsymbol{V}_4$?

(vi) Still in Minkowski spacetime, we define the dual $*\boldsymbol{\Psi}$ *of a basis 2-form* $\boldsymbol{\Psi}$ *by*

$$\boldsymbol{\Psi} \wedge *\boldsymbol{\Psi} = \pm\,\boldsymbol{V}_4$$

in which we choose minus (–) *if* $\boldsymbol{\Psi}$ *contains* **d***t, and we choose plus* (+) *otherwise. (Note that this rule encompasses the rule in the previous part as a special case.) Deduce that*

$*(\boldsymbol{d}x \wedge \boldsymbol{d}t) = -\boldsymbol{d}y \wedge \boldsymbol{d}z,\qquad *(\boldsymbol{d}y \wedge \boldsymbol{d}z) = \boldsymbol{d}x \wedge \boldsymbol{d}t$

$*(\boldsymbol{d}y \wedge \boldsymbol{d}t) = -\boldsymbol{d}z \wedge \boldsymbol{d}x \qquad *(\boldsymbol{d}z \wedge \boldsymbol{d}x) = \boldsymbol{d}y \wedge \boldsymbol{d}t$

$*(\boldsymbol{d}z \wedge \boldsymbol{d}t) = -\boldsymbol{d}x \wedge \boldsymbol{d}y \qquad *(\boldsymbol{d}x \wedge \boldsymbol{d}y) = \boldsymbol{d}z \wedge \boldsymbol{d}t$

$\boldsymbol{\Psi}$ is found within the parens on the LHS of each equation. Calculate $\boldsymbol{\Psi} \wedge *\boldsymbol{\Psi} = \pm\boldsymbol{V}_4$ to validate the equations:

$\star(\boldsymbol{d}x\wedge\boldsymbol{d}t)=-\boldsymbol{d}y\wedge\boldsymbol{d}z$
$\boldsymbol{\Psi}\wedge\star\boldsymbol{\Psi}=(\boldsymbol{d}x\wedge\boldsymbol{d}t)\wedge(-\boldsymbol{d}y\wedge\boldsymbol{d}z)$
$=\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}t\wedge\boldsymbol{d}z=-\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t=-\boldsymbol{V}_4$
$\boldsymbol{\Psi}$ contains **d**t; two swaps

$\star(\boldsymbol{d}y\wedge\boldsymbol{d}t)=-\boldsymbol{d}z\wedge\boldsymbol{d}x$
$\boldsymbol{\Psi}\wedge\star\boldsymbol{\Psi}=(\boldsymbol{d}y\wedge\boldsymbol{d}t)\wedge(-\boldsymbol{d}z\wedge\boldsymbol{d}x)=-\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}t\wedge(-\boldsymbol{d}z)$
$=-\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t=-\boldsymbol{V}_4$
$\boldsymbol{\Psi}$ contains **d**t; two swaps

$\star(\boldsymbol{d}z\wedge\boldsymbol{d}t)=-\boldsymbol{d}x\wedge\boldsymbol{d}y$
$\boldsymbol{\Psi}\wedge\star\boldsymbol{\Psi}=(\boldsymbol{d}z\wedge\boldsymbol{d}t)\wedge(-\boldsymbol{d}x\wedge\boldsymbol{d}y)=-\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t=-\boldsymbol{V}_4$
$\boldsymbol{\Psi}$ contains **d**t; four swaps

$\star(\boldsymbol{d}y\wedge\boldsymbol{d}z)=\boldsymbol{d}x\wedge\boldsymbol{d}t$
$\boldsymbol{\Psi}\wedge\star\boldsymbol{\Psi}=(\boldsymbol{d}y\wedge\boldsymbol{d}z)\wedge(\boldsymbol{d}x\wedge\boldsymbol{d}t)=\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t=\boldsymbol{V}_4$
$\boldsymbol{\Psi}$ does not contain **d**t; two swaps

$\star(\boldsymbol{d}z\wedge\boldsymbol{d}x)=\boldsymbol{d}y\wedge\boldsymbol{d}t$
$\boldsymbol{\Psi}\wedge\star\boldsymbol{\Psi}=(\boldsymbol{d}z\wedge\boldsymbol{d}x)\wedge(\boldsymbol{d}y\wedge\boldsymbol{d}t)=\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t=\boldsymbol{V}_4$
$\boldsymbol{\Psi}$ does not contain **d**t; two swaps

$\star(\boldsymbol{d}x\wedge\boldsymbol{d}y)=\boldsymbol{d}z\wedge\boldsymbol{d}t$
$\boldsymbol{\Psi}\wedge\star\boldsymbol{\Psi}=(\boldsymbol{d}x\wedge\boldsymbol{d}y)\wedge(\boldsymbol{d}z\wedge\boldsymbol{d}t)=\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t=\boldsymbol{V}_4$
$\boldsymbol{\Psi}$ does not contain **d**t; no swaps

The calculations support the equations.

(vii) Verify that for 2-forms in Minkowski spacetime, $\star\star=-1$*....*

$\star\star(\boldsymbol{d}x\wedge\boldsymbol{d}t)=\star(-\boldsymbol{d}y\wedge\boldsymbol{d}z)=-\boldsymbol{d}x\wedge\boldsymbol{d}t$ by linearity of $\star$.
$\star\star(\boldsymbol{d}y\wedge\boldsymbol{d}t)=\star(-\boldsymbol{d}z\wedge\boldsymbol{d}x)=-\boldsymbol{d}y\wedge\boldsymbol{d}t$ by linearity of $\star$.
$\star\star(\boldsymbol{d}z\wedge\boldsymbol{d}t)=\star(-\boldsymbol{d}x\wedge\boldsymbol{d}y)=-\boldsymbol{d}z\wedge\boldsymbol{d}t$ by linearity of $\star$.
$\star\star(\boldsymbol{d}y\wedge\boldsymbol{d}z)=\star(\boldsymbol{d}x\wedge\boldsymbol{d}t)=-\boldsymbol{d}y\wedge\boldsymbol{d}z$.
$\star\star(\boldsymbol{d}z\wedge\boldsymbol{d}x)=\star(\boldsymbol{d}y\wedge\boldsymbol{d}t)=-\boldsymbol{d}z\wedge\boldsymbol{d}x$
$\star\star(\boldsymbol{d}x\wedge\boldsymbol{d}y)=\star(\boldsymbol{d}z\wedge\boldsymbol{d}t)=-\boldsymbol{d}x\wedge\boldsymbol{d}y$

This confirms that $** = -1$.

(viii) Recall that the Faraday 2-form is given by (34.22):

$$\boldsymbol{F} = \textbf{Faraday} = \boldsymbol{\epsilon} \wedge \boldsymbol{d}t + \boldsymbol{B}.$$

Use the previous parts to deduce that the Maxwell 2-form (34.26) is indeed the Hodge dual of the Faraday 2-form.

$$*\boldsymbol{F} = \textbf{Maxwell} = \boldsymbol{\beta} \wedge \boldsymbol{d}t - \boldsymbol{E}.$$

$$\begin{aligned} &*(\boldsymbol{d}x \wedge \boldsymbol{d}t) = -\boldsymbol{d}y \wedge \boldsymbol{d}z, && *(\boldsymbol{d}y \wedge \boldsymbol{d}z) = \boldsymbol{d}x \wedge \boldsymbol{d}t \\ &*(\boldsymbol{d}y \wedge \boldsymbol{d}t) = -\boldsymbol{d}z \wedge \boldsymbol{d}x && *(\boldsymbol{d}z \wedge \boldsymbol{d}x) = \boldsymbol{d}y \wedge \boldsymbol{d}t \\ &*(\boldsymbol{d}z \wedge \boldsymbol{d}t) = -\boldsymbol{d}x \wedge \boldsymbol{d}y && *(\boldsymbol{d}x \wedge \boldsymbol{d}y) = \boldsymbol{d}z \wedge \boldsymbol{d}t \end{aligned}$$

$$\begin{aligned} \boldsymbol{F} &= \left(E_x\, \boldsymbol{d}x + E_y\, \boldsymbol{d}y + E_z\, \boldsymbol{d}z\right) \wedge \boldsymbol{d}t + \left[B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y)\right] \\ &= \left[E_x(\boldsymbol{d}x \wedge \boldsymbol{d}t) + E_y(\boldsymbol{d}y \wedge \boldsymbol{d}t) + E_z(\boldsymbol{d}z \wedge \boldsymbol{d}t)\right] + \left[B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y)\right] \end{aligned}$$

$$*\boldsymbol{F} = \left[*E_x(\boldsymbol{d}x \wedge \boldsymbol{d}t) + *E_y(\boldsymbol{d}y \wedge \boldsymbol{d}t) + *E_z(\boldsymbol{d}z \wedge \boldsymbol{d}t)\right] + \left[*B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + *B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + *B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y)\right]$$

$$\begin{aligned} &= \left[E_x(-\boldsymbol{d}y \wedge \boldsymbol{d}z) + E_y(-\boldsymbol{d}z \wedge \boldsymbol{d}x) + E_z(-\boldsymbol{d}x \wedge \boldsymbol{d}y)\right] + \left[B_x(\boldsymbol{d}x \wedge \boldsymbol{d}t) + B_y(\boldsymbol{d}y \wedge \boldsymbol{d}t) + B_z(\boldsymbol{d}z \wedge \boldsymbol{d}t)\right] = \\ &= -\left[E_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + E_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + E_z(\boldsymbol{d}x \wedge \boldsymbol{d}y)\right] + \left[B_x\, \boldsymbol{d}x + B_y\, \boldsymbol{d}y + B_z\, \boldsymbol{d}z\right] \wedge \boldsymbol{d}t \\ &= -\boldsymbol{E} + \boldsymbol{\beta} \wedge \boldsymbol{d}t = \boldsymbol{\beta} \wedge \boldsymbol{d}t - \boldsymbol{E} = \textbf{Maxwell} \end{aligned}$$

(ix) Likewise, verify that the dual of the spacetime current 1-form $\boldsymbol{J} = -\rho\, \boldsymbol{d}t + \underline{\boldsymbol{j}}$*, is indeed*

$*\boldsymbol{J} = -\rho \boldsymbol{V}_3 + [\textit{flux 2 - form of } \underline{\boldsymbol{j}}] \wedge \boldsymbol{d}t.$ (**j** is underlined)

$$\begin{aligned} *\boldsymbol{J} &= *(-\rho \boldsymbol{d}t) + *\underline{\boldsymbol{j}} \\ &= -\rho\,(\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z) + *[j_1\, \boldsymbol{d}x + j_2\, \boldsymbol{d}y + j_3\, \boldsymbol{d}z] && \text{[from (v) and p. 402]} \\ &= -\rho\, \boldsymbol{V}_3 + [j_1\, \boldsymbol{d}y \wedge \boldsymbol{d}z \wedge \boldsymbol{d}t + j_2\, \boldsymbol{d}z \wedge \boldsymbol{d}x \wedge \boldsymbol{d}t + j_3\, \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}t)] && \text{[from (v)]} \\ &= -\rho\, \boldsymbol{V}_3 + [\text{flux 2 - form of } \underline{\boldsymbol{j}}] \wedge \boldsymbol{d}t \end{aligned}$$