

Needham, VDGf, Chapter 39, Exercise 14, p. 467.

G. Palmer, August 3, 2026, 6:20 pm, PT

14. *Vector Calculus Identities via Forms. Use Forms to derive the following identities of Vector Calculus:*

$$(i) \nabla \bullet [\underline{\varphi} \times \nabla f] = [\nabla \times \underline{\varphi}] \bullet \nabla f$$

Note that $\mathbf{d}\underline{\varphi} \rightleftharpoons [\nabla \times \underline{\varphi}]$ by (36.15) and $\mathbf{d}\mathbf{d}f = 0$.

$$\begin{aligned} \nabla \bullet [\underline{\varphi} \times \nabla f] &\rightleftharpoons \mathbf{d}(\underline{\varphi} \wedge \mathbf{d}f) = \mathbf{d}\underline{\varphi} \wedge \mathbf{d}f - \underline{\varphi} \wedge \mathbf{d}\mathbf{d}f \\ &= \mathbf{d}\underline{\varphi} \wedge \mathbf{d}f \rightleftharpoons [\nabla \times \underline{\varphi}] \bullet \nabla f \end{aligned}$$

$$(ii) \nabla \bullet [\underline{\varphi} \times \underline{\psi}] = [\nabla \times \underline{\varphi}] \bullet \underline{\psi} - \underline{\varphi} \bullet [\nabla \times \underline{\psi}]$$

Note that $\mathbf{d}\underline{\psi} \rightleftharpoons [\nabla \bullet \underline{\psi}]$ by (36.16).

$$\begin{aligned} \nabla \bullet [\underline{\varphi} \times \underline{\psi}] &\rightleftharpoons \mathbf{d}(\underline{\varphi} \wedge \underline{\psi}) = \mathbf{d}\underline{\varphi} \wedge \underline{\psi} - \underline{\varphi} \wedge \mathbf{d}\underline{\psi} \\ &\rightleftharpoons [\nabla \times \underline{\varphi}] \bullet \underline{\psi} - \underline{\varphi} \bullet [\nabla \bullet \underline{\psi}] \end{aligned}$$