Needham, VDGF, Chapter 39, Exercise 13, p. 467.
G. Palmer, August 3, 2026, 5 pm, PT

*13. Closed and Exact Forms. If you did not do so earlier, prove each of the following facts:*

*(i) If* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\Phi}$ *are closed, then* $\boldsymbol{\gamma} \wedge \boldsymbol{\Phi}$ *is closed, too.*

If $\boldsymbol{\gamma}$ and $\boldsymbol{\Phi}$ are closed, then $\mathbf{d}\boldsymbol{\gamma} = \mathbf{d}\boldsymbol{\Phi} = 0$. Show that $\boldsymbol{d}(\boldsymbol{\gamma} \wedge \boldsymbol{\Phi}) = 0$.

$$\boldsymbol{d}(\boldsymbol{\gamma} \wedge \boldsymbol{\Phi}) = \mathbf{d}\boldsymbol{\gamma} \wedge \boldsymbol{\Phi} + (-1)^{\deg\gamma}\, \boldsymbol{\gamma} \wedge \mathbf{d}\boldsymbol{\Phi}$$

$$= \mathbf{0} \wedge \boldsymbol{\Phi} + (-1)^{\deg\gamma}\, \boldsymbol{\gamma} \wedge 0 = 0 + 0 = 0$$

as was to be shown.

*(ii) If* $\boldsymbol{\gamma}$ *is closed, then* $\boldsymbol{\gamma} \wedge \boldsymbol{d\Phi}$ *is closed for all* $\boldsymbol{\Phi}$.

If $\boldsymbol{\gamma}$ is closed, then $\mathbf{d}\boldsymbol{\gamma} = 0$.

$$\boldsymbol{d}(\boldsymbol{\gamma} \wedge \mathbf{d}\boldsymbol{\Phi}) = \mathbf{d}\boldsymbol{\gamma} \wedge \mathbf{d}\boldsymbol{\Phi} + (-1)^{\deg\gamma}\, \boldsymbol{\Phi} \wedge \boldsymbol{d}(\mathbf{d}\boldsymbol{\Phi})$$

$$= 0 \wedge \mathbf{d}\boldsymbol{\Phi} + (-1)^{\deg\gamma}\, \boldsymbol{\Phi} \wedge 0 = 0 + 0 = 0$$

as was to be shown.

*(iii) If degree* $\boldsymbol{\Phi}$ *is even, then* $\boldsymbol{\Phi} \wedge \boldsymbol{d\Phi}$ *is closed. Hint: See (35.4).*

If degree $\boldsymbol{\Phi}$ is even, then $\mathbf{d}\boldsymbol{\Phi}$ is odd, so $\mathbf{d}\boldsymbol{\Phi} \wedge \mathbf{d}\boldsymbol{\Phi} = 0$. Show that $\boldsymbol{d}(\boldsymbol{\Phi} \wedge \mathbf{d}\boldsymbol{\Phi}) = 0$.

$$\boldsymbol{d}(\boldsymbol{\Phi} \wedge \mathbf{d}\boldsymbol{\Phi}) = \boldsymbol{\Phi} \wedge \mathbf{d}(\mathbf{d}\boldsymbol{\Phi}) + (-1)^{2n}\, \mathbf{d}\boldsymbol{\Phi} \wedge \mathbf{d}\boldsymbol{\Phi}$$

$$= \boldsymbol{\Phi} \wedge 0 + (-1)^{2n}\, 0 = 0 + 0 = 0$$

as was to be shown.