Needham, VDGF, Chapter 39, Exercise 12, p. 467.

G. Palmer, August 3, 2026, 1:50 pm, PT

***12. Closed and Exact Forms***. *Continuing with the Forms of Exercise 11, compute each of the following external derivatives, (A) directly; (B) using the Leibniz product rule, (36.8).*

$\boldsymbol{\varphi} = 2\,x\boldsymbol{d}x + 2\,y\boldsymbol{d}y + 2\,z\boldsymbol{d}z$

$\boldsymbol{\chi} = \mathrm{xy}\boldsymbol{d}z$

$\boldsymbol{\psi} = x\boldsymbol{d}y \wedge \boldsymbol{d}z + y\boldsymbol{d}x \wedge \boldsymbol{d}z$

(i) $\boldsymbol{d}(\boldsymbol{\chi} \wedge \boldsymbol{\chi})$.

Directly:

$\boldsymbol{d}(\boldsymbol{\chi} \wedge \boldsymbol{\chi}) = 0$ because $\boldsymbol{\chi} = \mathrm{xy}\boldsymbol{d}z$ is odd (a 1-form).

$\boldsymbol{d}(\mathrm{xy}\mathbf{dz} \wedge \mathrm{xy}\mathbf{dz}) = \boldsymbol{d}\left(x^2\,y^2\right)\boldsymbol{d}z \wedge \boldsymbol{d}z = 0$ because $\boldsymbol{d}z \wedge \boldsymbol{d}z = 0$.

Leibniz:

$\boldsymbol{d}(\boldsymbol{\chi} \wedge \boldsymbol{\chi}) = \mathbf{d}\boldsymbol{\chi} \wedge \boldsymbol{\chi} + (-1)^1\,\boldsymbol{\chi} \wedge \mathbf{d}\boldsymbol{\chi} \mathbf{=} \mathbf{d}\boldsymbol{\chi} \wedge \boldsymbol{\chi} - \boldsymbol{\chi} \wedge \mathbf{d}\boldsymbol{\chi}$

$= \boldsymbol{\psi} \wedge \boldsymbol{\chi} \mathbf{-} \boldsymbol{\chi} \wedge \boldsymbol{\psi} = \boldsymbol{\psi} \wedge \boldsymbol{\chi} \mathbf{+} \boldsymbol{\psi} \wedge \boldsymbol{\chi} \mathbf{= 2}\,\boldsymbol{\psi} \wedge \boldsymbol{\chi}$

$= 2\,(x\boldsymbol{d}y \wedge \boldsymbol{d}z + y\boldsymbol{d}x \wedge \boldsymbol{d}z) \wedge \mathrm{xy}\boldsymbol{d}z = 0$ because $\boldsymbol{d}z \wedge \boldsymbol{d}z = 0$

(ii) $\boldsymbol{d}(\boldsymbol{\varphi} \wedge \boldsymbol{\chi})$.

Directly:

$\boldsymbol{d}([2\,x\boldsymbol{d}x + 2\,y\boldsymbol{d}y + 2\,z\boldsymbol{d}z] \wedge \mathrm{xy}\boldsymbol{d}z) = \boldsymbol{d}(2\,x\boldsymbol{d}x \wedge \mathrm{xy}\boldsymbol{d}z) + \boldsymbol{d}(2\,y\boldsymbol{d}y \wedge \mathrm{xy}\boldsymbol{d}z) + \boldsymbol{d}(2\,z\boldsymbol{d}z \wedge \mathrm{xy}\boldsymbol{d}z)$

$= \boldsymbol{d}(2\,x\boldsymbol{d}x \wedge \mathrm{xy}\boldsymbol{d}z) + \boldsymbol{d}(2\,y\boldsymbol{d}y \wedge \mathrm{xy}\boldsymbol{d}z) = \boldsymbol{d}\left(2\,x^2\,y\boldsymbol{d}x \wedge \boldsymbol{d}z\right) + \boldsymbol{d}\left(2\,\mathrm{xy}^2\,\boldsymbol{d}y \wedge \boldsymbol{d}z\right)$

Because taking the partial of a form component introduces a basis and wedge $\boldsymbol{d}_{(x,y,z)}\wedge$, one can only take $\partial_y f$ of $2\,x^2\,y\boldsymbol{d}x \wedge \boldsymbol{d}z$ and only $\partial_x f$ of $2\,\mathrm{xy}^2\,\boldsymbol{d}y \wedge \boldsymbol{d}z$.

$= 2\,x^2\,d\mathbf{y} \wedge \boldsymbol{d}x \wedge \boldsymbol{d}z + 2\,y^2\,\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = -2\,x^2\,\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z + 2\,y^2\,\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z$

$\boldsymbol{d}(\boldsymbol{\varphi} \wedge \boldsymbol{\chi}) = 2\left(y^2 - x^2\right)\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z$

Leibniz: We know from Exercise 11, part (i) that $\mathbf{d}\boldsymbol{\varphi} = 0$ and from part (ii) that $\boldsymbol{\psi} = \mathbf{d}\boldsymbol{\chi} = y\boldsymbol{d}x \wedge \boldsymbol{d}z + x\boldsymbol{d}y \wedge \boldsymbol{d}z$.

$$\boldsymbol{d}(\boldsymbol{\varphi} \wedge \boldsymbol{\chi}) = \mathbf{d}\boldsymbol{\varphi} \wedge \boldsymbol{\chi} - \boldsymbol{\varphi} \wedge \mathbf{d}\boldsymbol{\chi} = -\boldsymbol{\varphi} \wedge \mathbf{d}\boldsymbol{\chi}$$

$$= -(2\,x\boldsymbol{d}x + 2\,y\boldsymbol{d}y + 2\,z\boldsymbol{d}z) \wedge (x\boldsymbol{d}y \wedge \boldsymbol{d}z + y\boldsymbol{d}x \wedge \boldsymbol{d}z)$$

Excluding multiples that contain vanishing wedge combinations,

$$= -(2\,x\boldsymbol{d}x \wedge x\boldsymbol{d}y \wedge \boldsymbol{d}z + 2\,y\boldsymbol{d}y \wedge y\boldsymbol{d}x \wedge \boldsymbol{d}z)$$

$$= -\left(2\,x^2\,\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z - 2\,y^2\,\boldsymbol{d}x \wedge y\boldsymbol{d}y \wedge \boldsymbol{d}z\right)$$

$$= 2\left(y^2 - x^2\right)\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z$$

(iii) $\boldsymbol{d}(\boldsymbol{\varphi} \wedge \boldsymbol{\Psi})$.

Directly:

$$\boldsymbol{d}(\boldsymbol{\varphi} \wedge \boldsymbol{\Psi}) = \boldsymbol{d}[(2\,x\boldsymbol{d}x + 2\,y\boldsymbol{d}y + 2\,z\boldsymbol{d}z) \wedge (x\boldsymbol{d}y \wedge \boldsymbol{d}z + y\boldsymbol{d}x \wedge \boldsymbol{d}z)]$$

$$= \boldsymbol{d}(2\,x\boldsymbol{d}x \wedge x\boldsymbol{d}y \wedge \boldsymbol{d}z + 2\,y\boldsymbol{d}y \wedge y\boldsymbol{d}x \wedge \boldsymbol{d}z)$$

$$= \boldsymbol{d}\left(2\,x^2\,\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z + 2\,y^2\,\boldsymbol{d}y \wedge \boldsymbol{d}x \wedge \boldsymbol{d}z\right) = 0$$

Because every **d**x, **d**y, and **d**z is annihilated by the external **d**.

Leibniz: $\boldsymbol{\varphi}$ and $\boldsymbol{\Psi}$ are both closed, so

$$\boldsymbol{d}(\boldsymbol{\varphi} \wedge \boldsymbol{\Psi}) = \mathbf{d}\boldsymbol{\varphi} \wedge \boldsymbol{\Psi} - \boldsymbol{\varphi} \wedge \mathbf{d}\boldsymbol{\Psi} = 0 \wedge \boldsymbol{\Psi} - \boldsymbol{\varphi} \wedge 0 = 0$$

(iv) $\boldsymbol{d}(\boldsymbol{\chi} \wedge \boldsymbol{\Psi})$

Directly:

$$\boldsymbol{d}(\boldsymbol{\chi} \wedge \boldsymbol{\Psi}) = \boldsymbol{d}[\mathrm{xy}\boldsymbol{d}z \wedge (x\boldsymbol{d}y \wedge \boldsymbol{d}z + y\boldsymbol{d}x \wedge \boldsymbol{d}z)]$$

$$= \boldsymbol{d}(0 + 0) = 0 \quad (\text{because } \boldsymbol{d}z \wedge \boldsymbol{d}z = 0)$$

Leibniz: From Exercise 11 (ii), $\boldsymbol{\Psi}$ is closed, and $\mathbf{d}\boldsymbol{\chi} = \boldsymbol{\Psi}$.

$$\boldsymbol{d}(\boldsymbol{\chi} \wedge \boldsymbol{\Psi}) = \mathbf{d}\boldsymbol{\chi} \wedge \boldsymbol{\Psi} - \boldsymbol{\chi} \wedge \mathbf{d}\boldsymbol{\Psi} = \boldsymbol{\Psi} \wedge \boldsymbol{\Psi} - \boldsymbol{\chi} \wedge 0 = 0 - 0 = 0$$