Needham, VDGF, Chapter 39, Exercise 11, p. 467.
G. Palmer, Aug 2, 2026, 4:15 pm, PT

11. Closed and Exact Forms*. In* $\mathbb{R}^3$*, let*

$$\boldsymbol{\varphi} = 2\,x\boldsymbol{d}x + 2\,y\boldsymbol{d}y + 2\,z\boldsymbol{d}z$$
$$\chi = xy\boldsymbol{d}z$$
$$\boldsymbol{\psi} = x\boldsymbol{d}y \wedge \boldsymbol{d}z + y\boldsymbol{d}x \wedge \boldsymbol{d}z$$

(i) Verify that $\boldsymbol{\varphi}$ *is closed, and explain this by showing that it is exact:* $\boldsymbol{\varphi} = \boldsymbol{d}r^2$*, where r is the distance from the origin.*

Refer to the spherical model in [35.1] and use the equations:

$$x = r\sin\phi\cos\theta \equiv r\,\mathrm{s}\phi\,\mathrm{c}\theta$$
$$y = r\sin\phi\sin\theta \equiv r\,\mathrm{s}\phi\,\mathrm{s}\theta$$
$$z = r\cos\phi \equiv r\,\mathrm{c}\phi$$

$$\boldsymbol{d}(x) = \boldsymbol{d}r\,\mathrm{s}\phi\,\mathrm{c}\theta + r\,\mathrm{c}\phi\,\boldsymbol{d}\phi\,\mathrm{c}\theta - r\,\mathrm{s}\phi\,\mathrm{s}\theta\,d\boldsymbol{\theta}$$
$$\boldsymbol{d}(y) = \boldsymbol{d}r\,\mathrm{s}\phi\,\mathrm{s}\theta + r\,\mathrm{c}\phi\,\boldsymbol{d}\phi\,\mathrm{s}\theta + r\,\mathrm{s}\phi\,\mathrm{c}\theta\,d\boldsymbol{\theta}$$
$$\boldsymbol{d}(z) = \boldsymbol{d}r\,\mathrm{c}\phi - r\,\mathrm{s}\phi\,\boldsymbol{d}\phi$$

$$\begin{aligned}\boldsymbol{\varphi} = 2\,\{&[r\,\mathrm{s}\phi\,\mathrm{c}\theta\,(\boldsymbol{d}r\,\mathrm{s}\phi\,\mathrm{c}\theta + r\,\mathrm{c}\phi\,\boldsymbol{d}\phi\,\mathrm{c}\theta - r\,\mathrm{s}\phi\,\mathrm{s}\theta\,\boldsymbol{d}\theta)]\\ &+[r\,\mathrm{s}\phi\,\mathrm{s}\theta\,(\boldsymbol{d}r\,\mathrm{s}\phi\,\mathrm{s}\theta + r\,\mathrm{c}\phi\,\boldsymbol{d}\phi\,\mathrm{s}\theta + r\,\mathrm{s}\phi\,\mathrm{c}\theta\,\boldsymbol{d}\theta)]\\ &+[\,r\,\mathrm{c}\phi\,(d\boldsymbol{r}\,\mathrm{c}\phi - r\,\mathrm{s}\phi\,\boldsymbol{d}\phi)]\}\end{aligned}$$

$$\begin{aligned}&= 2\left\{r\,\boldsymbol{d}r\left(s^2\phi\,c^2\theta + s^2\phi\,s^2\theta + c^2\phi\right) + r^2[(\mathrm{s}\phi\,\mathrm{c}\theta\,\mathrm{c}\phi\,\boldsymbol{d}\phi\,\mathrm{c}\theta + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\phi\,\boldsymbol{d}\phi\,\mathrm{s}\theta - \mathrm{c}\phi\,\mathrm{s}\phi\,\boldsymbol{d}\phi)\right.\\ &+\mathrm{s}\phi\,\mathrm{c}\theta\,(-\mathrm{s}\phi\,\mathrm{s}\theta\,\boldsymbol{d}\theta) + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\phi\,\mathrm{c}\theta\,\boldsymbol{d}\theta)]\end{aligned}$$ Collect by **d**r, **d**ϕ, **d**θ.

$$\begin{aligned}&= 2\left\{r\,\boldsymbol{d}r\left(s^2\phi + c^2\phi\right) + r^2\left[\left(\mathrm{s}\phi\,\mathrm{c}\phi\;c^2\theta + \mathrm{s}\phi\,\mathrm{c}\phi\,s^2\theta - \mathrm{c}\phi\,\mathrm{s}\phi\right)\boldsymbol{d}\phi\right.\right. \quad \left(c^2 + s^2 = 1\right)\\ &+(-\mathrm{s}\phi\,\mathrm{c}\theta\,\mathrm{s}\phi\,\mathrm{s}\theta + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\phi\,\mathrm{c}\theta)\,\boldsymbol{d}\theta]\}\end{aligned}$$

$$= 2\left\{r\,\boldsymbol{d}r + r^2[0\cdot\boldsymbol{d}\phi + 0\cdot\boldsymbol{d}\theta]\right\}$$

$$= 2\,r\,\boldsymbol{d}r = d\left(r^2\right) = \boldsymbol{d}r^2$$

This is what we wanted. But one can also verify that $\boldsymbol{\varphi}$ is closed by taking the derivative of 2r **d**r.

$$\mathbf{d}\boldsymbol{\varphi} = \boldsymbol{d}(2\,r\boldsymbol{d}r) = 2\,\boldsymbol{d}(r\boldsymbol{d}r) = 2\,[\boldsymbol{d}r \wedge \boldsymbol{d}r \;+\; r\,\boldsymbol{d}(\boldsymbol{d}r)] = 0 + 0 \quad \text{(Leibniz)}$$

$\boldsymbol{d}(2\,r\boldsymbol{d}r) \;=\; 0$, because “every form is annihilated by two applications of **d**.” (p. 396). The same principle applies to $\boldsymbol{d}\varphi = 2\,x\boldsymbol{d}x + 2\,y\boldsymbol{d}y + 2\,z\boldsymbol{d}z$.

(ii) Verify that $\boldsymbol{\Psi}$ *is closed and explain this by showing that it is exact:* $\boldsymbol{\Psi} = \boldsymbol{d}\boldsymbol{\mathcal{X}}$ *.*

d requires one to sum the partials and insert an initial "$\boldsymbol{d}_{x,y,z}\wedge$" form for each partial into the basis, canceling results with duplicate **d** forms.

$$\mathbf{d}\boldsymbol{\mathcal{X}} = \boldsymbol{d}(\mathrm{xy}\boldsymbol{d}z) = y\boldsymbol{d}x \wedge \boldsymbol{d}z + x\boldsymbol{d}y \wedge \boldsymbol{d}z + 0 = \boldsymbol{\psi}$$

A form is exact if it is a derivative. Since $\boldsymbol{\psi}$ is a derivative, it is closed. $\mathbf{d}\boldsymbol{\psi} = 0$

$$\mathbf{d}\boldsymbol{\psi} = \boldsymbol{d}(y\boldsymbol{d}x \wedge \boldsymbol{d}z) + \boldsymbol{d}(x\boldsymbol{d}y \wedge \boldsymbol{d}z) = (0 + \boldsymbol{d}y \wedge \boldsymbol{d}x \wedge \boldsymbol{d}z + 0) + (\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z + 0 + 0)$$

$$= \boldsymbol{d}y \wedge \boldsymbol{d}x \wedge \boldsymbol{d}z + \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = -\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z + \boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z = 0$$